Topology Preliminary Exam

Spring 2020

1. Show that if a subspace A of a space X is connected then so is its closure $\overline{A}$.

2. (a) If X consists of n points with the discrete topology show that $\mathrm{Map}(X, Y)$ is homeomorphic to $\underbrace{Y \times Y \times \ldots \times Y}_{n}$.

   (b) If X has the trivial topology and Y is Hausdorff, identify the space $\mathrm{Map}(X, Y)$.

3. Consider the relation on $(0, \infty)$ given by $x \sim y$ if there is an integer n such that $x/y = 2^n$. Prove that $\sim$ is an equivalence relation and show that the quotient space $(0, \infty)_{/\sim}$ is homeomorphic to S^1.

4. Show that if G is a locally compact and Hausdorff topological group and H is a subgroup, then G/H is locally compact.

5. Let $p\colon \bar{X} \to X$ be a covering map and let $A \subseteq X$ be a connected subspace. Let $\bar{A}$ be a component of $p^{-1}A \subseteq \bar{X}$.

   (a) Show that the restriction $p\colon \bar{A} \to A$ is a covering map.

   (b) Assume that $\bar{X}$ is the universal cover of X. Is $\bar{A}$ the universal cover of A?

6. Let $f : D^2 \to \mathbb{R}^2$ be a continuous map. Suppose $f|_{S^1}$ factors through $\mathbb{R}^2 \setminus 0$. That is, there is a commutative diagram

$$\begin{array}{ccc} D^2 & \xrightarrow{f} & \mathbb{R}^2 \\ \uparrow & & \uparrow \\ S^1 & \xrightarrow{g} & \mathbb{R}^2 \setminus 0 \end{array}$$

   If g induces a nonzero map

$$\pi_1(S^1) \to \pi_1(\mathbb{R}^2 \setminus 0),$$

   show there exists a point $x \in D^2$ such that $f(x) = 0$.

7. Let $T_2 = T \# T$ be the connected sum of two copies of the torus $T = S^1 \times S^1$. Show that T_2 is not a covering space of T. (You can use facts about fundamental groups of surfaces.)

8. Let X be a path-connected space with fundamental group $\pi_1(X) \cong \Sigma_3$, the symmetric group of order 6. Let $\alpha\colon S^1 \to X$ be a loop of order two, and let $Y = X \cup_\alpha D^2$ be the space obtained by attaching a disk to X by gluing the boundary of D^2 to X using the attaching map α. Find $\pi_1(Y)$.