Topology Preliminary Exam
June, 2021

1. Let x be a point in S^1.

a. Prove that $S^1 - \{x\}$ (as a subspace of S^1) is homeomorphic to $\mathbb{R}$ (with the Euclidean topology).

b. Prove that no homeomorphism $S^1 - \{x\} \cong \mathbb{R}$ extends to a continuous map $S^1 \to \mathbb{R}$.

2. Assume $p\colon Y \to X$ is a quotient map. Further, assume that X is connected and that, for each $x \in X$, the subspace $p^{-1}(x) \subseteq Y$ is connected. Prove that Y is connected.

3. a. Let X and Y be locally compact spaces and assume that Y is Hausdorff. Show that $C(X, Y)$, the set of continuous functions from X to Y endowed with the compact open topology, is Hausdorff.

b. Fix Y and assume that there exists a nonempty space X with the property that $C(X, Y)$ is Hausdorff. Show that Y must be Hausdorff.

4. Assume that X is a Hausdorff space in which no singleton is open.

a. Let U be a nonempty open subset of X and let $x \in X$. Prove that there exists $V \subseteq U$ such that $x \notin \bar{V}$.

b. Further assume that X is compact and let $\mathbb{N}$ be the set of natural numbers with the discrete topology. Prove that there is no surjective continous map $f\colon \mathbb{N} \to X$. (Hint find a nested sequence of open sets whose closures miss more and more points in the image of f.)

5. Show that the projection map $X \times Y \to X$ is a covering map when Y has the discrete topology.

6. Let $A \subset \mathbb{R}^2$ be the unit circle and let X be the union of A with the line segment $[-1, 1] \times \{0\}$. Show that A is a retract of X but not a deformation retract of X.

7. Let G be a topological group with multiplication μ and identity e. (Also assume G is path connected and locally path connected.) If $(\tilde{G}, p)$ is a cover of G and $\tilde{e} \in \tilde{G}$ satisfies $p(\tilde{e}) = e$, show that there is a unique continuous multiplication on $\tilde{G}$ that commutes with p and has $\tilde{e}$ as the unit.

8. Construct a non-normal covering space of $S^1 \vee S^1$.