Topology Preliminary Exam
June 2022

1. Let X be a topological space, and A and B compact subspaces.

(a) Show that $A \cup B$ is compact.

(b) Show that if X is Hausdorff, then $A \cap B$ is compact.

(c) Give a counterexample to part (b) in the case when X is not Hausdorff.

2. Let X and Y be locally compact Hausdorff spaces with one-point compactifications $\tilde{X}$ and $\tilde{Y}$. Show that there is a quotient map $\tilde{X} \times \tilde{Y} \to \widetilde{X \times Y}$, where $\widetilde{X \times Y}$ is the one-point compactification of $X \times Y$.

3. Let I be the closed interval and let $C(I, X)$ be the set of continuous functions from I to X equipped with the compact open topology. Show that $C(I, X)$ is path-connected if and only if X is path-connected.

4. Show that the quotient map $q\colon S^2 \to \mathbb{R}P^2$ is not homotopic to a constant map.

5. Let $f\colon D^2 \to \mathbb{R}^2$ be a continuous map that leaves each point of the boundary circle S^1 fixed. Show $D^2 \subset f(D^2)$.

6. (a) Let A be a single circle in $\mathbb{R}^3$. Compute $\pi_1(\mathbb{R}^3 \setminus A)$. (Remember $\mathbb{R}^3 \setminus A$ is the complement of A in $\mathbb{R}^3$.)

(b) Let A and B be disjoint circles in $\mathbb{R}^3$. Compute $\pi_1(\mathbb{R}^3 \setminus (A \cup B))$.

(c) How does $\pi_1(\mathbb{R}^3 \setminus (A \cup B))$ change if the circles are linked?

7. Let X be the space obtained by beginning with S^2 and attaching two line segments between the poles.

(a) Calculate the fundamental group of X for any choice of base point.

(b) Describe the universal cover of X.

(c) Calculate the integral homology of X.

8. Suppose the two ends of $S^2 \times I$ are glued together via a map $S^2 \to S^2$ of degree d. (Recall that a map $S^n \to S^n$ is **degree** d if it induces the multiplication by d map on $H_n(S^n)$.) Use the Mayer-Vietoris sequence to calculate the homology of the resulting space X.