DEPARTMENT OF MATHEMATICS

Topology Preliminary Examination
June 5, 2023

Instructions: A necessary condition to pass this exam is to completely solve one point-set question and one of the more algebraic questions. An excellent exam will completely solve five problems and have some partial credit on the remaining questions.

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1) Suppose that $f\colon X \longrightarrow Y$ is a continuous map such that there exists a continuous map $g\colon Y \longrightarrow X$ such that the composition $f \circ g\colon Y \longrightarrow Y$ is the identity map. Show that f is a quotient map.

2) Suppose that X is compact and that points in Y are closed. Show that if $f\colon X \longrightarrow Y$ is a local homeomorphism, then, for any point $y \in Y$, the preimage $f^{-1}(y)$ is finite. If Y is also a connected Hausdorff space, show that f is surjective.

3) Let $f\colon W \longrightarrow X$ and $g\colon Y \longrightarrow Z$ be continuous. Let $\mathrm{Map}(X, Y)$ denote the set of continuous maps $X \longrightarrow Y$, equipped with the compact-open topology, and similarly for $\mathrm{Map}(W, Z)$. Show that the function $\phi\colon \mathrm{Map}(X, Y) \longrightarrow \mathrm{Map}(W, Z)$, defined by $\phi(h) = g \circ h \circ f$, is continuous.

4) Let $T^2 = S^1 \times S^1$ be the torus, and let x, y, and z be distinct points in T^2. Find the fundamental group of $X = T^2 - \{x, y, z\}$ (the complement of three points in T^2).

5) Recall that the real projective plane $\mathbb{RP}^2$ is pointed via a canonical inclusion $\mathbb{RP}^0 \hookrightarrow \mathbb{RP}^2$.

(a) Show that $\pi_1(\mathbb{RP}^2 \vee \mathbb{RP}^2)$ is isomorphic to $\pi_1(\mathbb{RP}^2) * \pi_1(\mathbb{RP}^2)$.

(b) Show that every map $\mathbb{RP}^2 \vee \mathbb{RP}^2 \to S^1$ is homotopic to a constant map.

6) Suppse that E and B are connected and locally path-connected. Let $p\colon E \longrightarrow B$ be a two-sheeted covering. Show that there is a free action of $G = C_2$, the group of order two, on E such that the quotient E/G is homeomorphic to B.

7) Let $S^2 \subset S^3 \subset S^4$ be equatorial inclusions.

(a) Compute the relative homology groups $\mathrm{H}_*(S^4, S^3)$.
(b) Compute the relative homology groups $\mathrm{H}_*(S^4, S^2)$.

8) (a) Let X be obtained from a torus $S^1 \times S^1$ by attaching a Mobius band via a homeomorphism from the boundary circle of the Mobius band to a circle $S^1 \times x_0$ in the torus. Use the Mayer-Vietoris sequence to compute the homology of X.
(b) Let Y be obtained from $\mathbb{RP}^2$ by attaching a Mobius band via a homeomorphism from the boundary circle of the Mobius band to the standard $\mathbb{RP}^1$ in $\mathbb{RP}^2$. Use the Mayer-Vietoris sequence to compute the homology of Y.