Topology Preliminary Exam
Summer 2024

On grading: A necessary condition to pass this exam is to completely solve one point-set question and one of the more algebraic questions. An excellent exam will completely solve five problems and have some partial credit on the remaining questions.

(1) Let $f\colon X \to X$ be a continuous function and suppose that X is Hausdorff (T_2).
   (a) Prove the graph of f is closed in $X \times X$.
   (b) Show the fixed points of f are a closed subset of X.

(2) Suppose $i\colon Z \to Y$ is an inclusion. Show the induced map

$$i_*\colon \mathrm{Map}(X, Z) \to \mathrm{Map}(X, Y)$$

is an injection and a homeomorphism onto its image.

(As a set $\mathrm{Map}(X, Z)$ is the continuous functions from X to Z. It is given the compact open topology.)

(3) Let $\mathbb{R}$ act on $\mathbb{R}^2$ by $t \cdot (x, y) = (x, y + tx)$.
   (a) Prove that the quotient by this action, with the quotient topology, is not Hausdorff, but is the union of two Hausdorff subspaces.
   (b) Show that the quotient is a T_1-space. That is, for every pair of distinct points, each is contained in a neighborhood not containing the other point.

(4) Show that there is no retraction $r : S^1 \times D^2 \to S^1 \times S^1$.

(5) Let $\alpha\colon S^1 \to S^1 \times S^1$ be given by $\alpha(z) = (z, (1, 0))$. Let X be the space given by attaching two disks to $S^1 \times S^1$. The first disk is attached along the image of α by the degree 3 map and the second is attached along the image of α by the degree 2 map. Compute the fundamental group of X.

(6) Find all 3-fold connected coverings of $\mathbb{R}P^2 \vee S^1$ up to isomorphism.

(7) Use Mayer-Vietoris to compute (for any X) the homology $H_*(X \times S^1)$ in terms of $H_*(X)$. Use this to compute the homology of the n-torus $T^n = S^1 \times \cdots \times S^1$.

(8) Let $X = S^1 \times S^1$ and $Y = S^1 \vee S^1 \vee S^2$ with universal covers $\tilde{X}$ and $\tilde{Y}$, respectively.
   (a) Show that X and Y have isomorphic homology groups.
   (b) Show that the homology of $\tilde{X}$ differs from the homology $\tilde{Y}$.