1. Introduction

The notion of trace is ubiquitous in mathematics. A recurring theme is that traces turn complicated objects into simpler ones, discarding much of the information in the original object but retaining enough to say something useful about it. Many familiar invariants arise as traces of identity maps; for example, the trace of the identity map on a real or complex vector space is the vector space's dimension. While I am primarily interested in traces arising in algebraic settings, the framework I use to study traces is so general that it also has topological applications, such as the Euler characteristic (i.e. vertices minus edges plus faces).

One reason traces are so useful is that they are compatible with many kinds of structure. For example, the following identities hold for matrices A and B:

$$(1) \quad \mathrm{tr}(A+B) = \mathrm{tr}(A) + \mathrm{tr}(B) \qquad \text{and} \qquad (2) \quad \mathrm{tr}(A \otimes B) = \mathrm{tr}(A) \cdot \mathrm{tr}(B).$$

These properties are not specific to linear algebra; they hold for vastly more general notions of trace. An example of Equation (2) comes from restriction and induction of group representations. A complex representation V of a group G can be viewed as a left module over the group ring $\mathbb{C}[G]$. Applying a generalized trace to the identity map on V recovers the group character, which is the function $G \to \mathbb{C}$ taking g to the (ordinary) trace of the linear map induced on V by the action of G.

If H is a subgroup of G, then representations of G can be restricted to representations of H, and representations of H can be induced up to representations of G. Expressing the restricted or induced representation as a tensor product with a certain bimodule over the group algebras of G and H has the following consequence:

Theorem 1.1 (Ponto)**.** *The classical formulas for the restricted and induced characters are examples of multiplicativity of trace (Equation (2)).*

Another property of traces that I study is their interaction with a "pairing" map which combines two objects A and B to produce a third object $\langle A, B\rangle$. In the motivating example of this pairing, one of the objects admits a notion of trace, but the other required me to define a dual notion of "cotrace." The resulting compatibility with structure is an "adjointness" between trace and cotrace:

$$(3) \quad \langle \mathrm{cotr}(A), B\rangle = \langle A, \mathrm{tr}(B)\rangle$$

Making this relationship precise (Theorem 2.1) and developing the requisite theory of cotraces has led to novel variations on Equation (2) and surprising connections between commutative algebra and representation theory.

2. Main Results

My work focuses on cotraces, which are in a certain sense dual to traces. Building on Ponto's work [Pon08, PS12], I described the structure necessary to support a notion of cotrace, and established properties of the cotrace analogous to those of the trace. Cotraces provide similar information to traces in some settings; for example, in the case of representations, a certain cotrace captures precisely the same information as the group character, which is an example of a trace. Other examples, however, are not adequately explained by traces, so the cotrace is a distinct and necessary component of this framework.

2.1. Applications of Cotraces to Commutative Algebra.

The first of these examples, and my original motivation for developing a notion of cotrace, is Joseph Lipman's [Lip87] "trace" and "cotrace" maps, which interact with each other through a pairing map. Lipman provides a more elementary development of Grothendieck's residue symbol [Har66] by reframing it in terms of Hochschild homology. However, Lipman acknowledges that this description of residues is not fully satisfactory, and he suspects that "there might well be a more fundamental approach to the subject, encompassing a great deal more than we have dealt with here" ([Lip87]).

I have offered a candidate for the more fundamental approach Lipman imagined, by repackaging his traces and cotraces in terms of Ponto's bicategorical traces [Pon08, PS12] and my bicategorical cotraces. In doing so, I have teased out the formal structure underlying Lipman's trace formulas, which includes (1) a notion of "coshadow" and "cotrace" dual to Ponto's shadows and traces and (2) an interplay between traces and cotraces making precise the relation informally described by Equation (3):

Theorem 2.1 (B.)**.** *Given suitable maps f, g, h, i involving objects A, A', B, B', C and (co)shadows $\langle\!\langle - \langle\!\langle$, $\langle\!\langle - \rangle\!\rangle$, $\langle\!\langle - \rangle\!\rangle$ with a pairing map $\langle\!\langle - \langle\!\langle \otimes \langle\!\langle - \rangle\!\rangle \xrightarrow{\rho} \langle\!\langle - \otimes - \rangle\!\rangle$, the following commutes:*

$$\begin{array}{ccccc}
\langle\!\langle A' \langle\!\langle \otimes \langle\!\langle B' \rangle\!\rangle & \xleftarrow{\mathrm{cotr}(f)\otimes 1} & \langle\!\langle A \langle\!\langle \otimes \langle\!\langle B' \rangle\!\rangle & \xrightarrow{1\otimes \mathrm{tr}(g)} & \langle\!\langle A \langle\!\langle \otimes \langle\!\langle B \rangle\!\rangle \\
\downarrow \rho & & & & \downarrow \rho \\
\langle\!\langle A' \otimes B' \rangle\!\rangle & \xrightarrow[\mathrm{tr}(h)]{} & \langle\!\langle C \rangle\!\rangle & \xleftarrow[\mathrm{tr}(i)]{} & \langle\!\langle A \otimes B \rangle\!\rangle
\end{array}$$

This generalizes Lipman's main results (Proposition 4.5.4 and Theorem 4.7.2 of [Lip87]).

Ponto's traces and my cotraces are both formulated in the language of category theory, which provides a way of describing common structure across different areas of mathematics. This abstract perspective is valuable because it often allows us to make mathematical constructs and theorems more accessible by extracting the core ideas from the technical details of their original presentation. Moreover, we are often able to prove vastly more general versions of these results and port them over to other mathematical contexts. For example, the tools that I have built to understand Lipman's work create a bridge to the theory of group representations and 2-representations.

2.2. **Applications of Cotraces to Representation Theory.** The character of a group representation can be obtained from either a trace or a cotrace. Just as the characters of restricted and induced representations are consequences of multiplicativity of the trace (Theorem 1.1), we can deduce those formulas from formal properties of the cotrace:

Proposition 2.2 (B.)**.** *The formulas for the restricted and (co)induced characters are examples of a property of cotraces analogous to Equation (2).*

Weakening the conditions defining a representation leads to "2-representations" and "2-characters." Ganter and Kapranov [GK06] proved an induction formula for 2-representations; Travis Wheeler and I have made progress toward understanding this result from the perspective of bicategorical (co)traces, starting by identifying their "categorical trace," a key ingredient in the 2-character:

Proposition 2.3 (B.)**.** *The "categorical trace" of* [GK06] *is an example of a coshadow.*

By pinning down the appropriate bicategorical setting for the work of Ganter and Kapranov and describing their 2-character in the framework of cotraces, we hope to show:

Conjecture 2.4. *The induction formula for 2-characters is an example of properties of the cotrace. Moreover, the 2-character can be computed with either a trace or a cotrace as a consequence of Theorem 2.1.*

The program of bicategorical shadows and traces aims to unify seemingly disparate pieces of mathematics underneath a common conceptual framework. By adding notions of coshadow and cotrace to this machinery, I have drawn Lipman's residues and (co)traces into this framework and opened the road to incorporating Ganter and Kapranov's 2-characters in a way that parallels the application of traces to ordinary group representations.

3. Future Work

In addition to reframing Ganter and Kapranov's induction formula with (co)traces, my work will proceed in several directions:

- In August I participated in a workshop organized as part of an NSF RTG grant at the University of Virginia; I worked with Nick Kuhn, Don Larsen, William Balderrama, and Andres Mejia to understand certain algebraic operations on modules arising in algebraic topology. Following the workshop, I wrote a library of Python code to perform calculations with these modules; the code has helped reveal patterns which have led to proofs about these operations. I plan to continue collaborating on this project.
- Bicategorical traces have applications to the study of fixed points, such as the Lefschetz-Hopf theorem and analogues for the Reidemeister trace. I plan to investigate whether cotraces have similar applications to fixed point theory.
- Lipman [Lip87] translated ideas from algebraic geometry into commutative algebra with Hochschild (co)homology. Having reinterpreted Lipman's (co)traces as bicategorical (co)traces, I plan to reverse-engineer this translation in order to apply these trace methods to the original algebro-geometric situation.

References

[GK06] Nora Ganter and Mikhail Kapranov, *Representation and character theory in 2-categories.* (cit. on p. 2).

[Har66] Robin Hartshorne, *Residues and duality*, vol. 20, Springer, 1966. (cit. on p. 1).

[Lip87] Joseph Lipman, *Residues and traces of differential forms via hochshild homology*, Contemporary Math. **61** (1987). (cit on pp. 1, 2).

[Pon08] Kate Ponto, *Fixed point theory and trace for bicategories*, Astérisque (2008). (cit. on p. 1).

[PS12] Kate Ponto and Michael Shulman, *Shadows and traces in bicategories*, Journal of Homotopy and Related Structures **8** (2012), no. 2, 151–200. (cit. on p. 1).