

YUE CAI

RESEARCH STATEMENT

I am interested in Algebraic Combinatorics. My research focuses on new expressions for q -analogues and the algebraic and topological implications behind them.

0 Introduction to q -analogues

The idea of q -analogues can be traced back to Euler in the 1700's who was studying q -series, especially specializations of theta functions. In general, a q -analogue of a combinatorial object is an expression parameterized by q which reduces to the cardinality of the object when we set $q = 1$. The theory of q -analogues has applications to the study of quantum calculus, quantum groups and many other algebraic and analytic fields. For example, let \mathfrak{S}_n be the set of all permutations on $\{1, 2, \dots, n\}$ and denote the *inversion number* of a permutation $\pi \in \mathfrak{S}_n$ to be $\text{inv}(\pi) = |\{(i, j) : i < j \text{ and } \pi_i > \pi_j\}|$. There are $n!$ permutations of length n , and the q -analogue $[n]_q! = [n]_q \cdot [n-1]_q \cdots [1]_q$ with $[k]_q = 1 + q + \cdots + q^{k-1}$ keeps track of the inversion numbers of the permutations, that is,

$$\sum_{\pi \in \mathfrak{S}_n} q^{\text{inv}(\pi)} = [n]_q!.$$

This is due to MacMahon [17]. In particular, setting $q = 1$ we obtain the classical factorial $n!$.

1 q -Stirling numbers of the second kind

A *set partition* on n elements $\{1, 2, \dots, n\}$ is a decomposition of this set into mutually disjoint nonempty sets called blocks. The q -Stirling numbers of the second kind are defined by

$$S_q[n, k] = S_q[n-1, k-1] + [k]_q \cdot S_q[n-1, k] \quad \text{for } 1 \leq k \leq n,$$

with the boundary conditions $S_q[n, k] = \delta_{n,0}$ and $S_q[0, k] = \delta_{0,k}$. Setting $q = 1$ gives the familiar Stirling number of the second kind $S(n, k)$ which enumerates the number of partitions of $\{1, 2, \dots, n\}$ with exactly k blocks. Set partitions can be encoded by *restricted growth word* or *RG-words*. Given a partition $\pi = B_1/B_2/\cdots/B_k$ of $\{1, 2, \dots, n\}$ with $\min(B_1) < \min(B_2) < \cdots < \min(B_k)$, define the *RG-word* $w = w_1 w_2 \cdots w_n$ by $w_i = j$ if the element i occurs in the j th block B_j of π . There is a long history of studying set partition statistics [12, 16, 20] and q -Stirling numbers [5, 9, 13, 18, 26].

Given an *RG-word* $w = w_1 w_2 \cdots w_n \in \mathcal{R}(n, k)$, we would like a statistic to generate the q -Stirling numbers of the second kind. Define the weight $\text{wt}(w) = q^{\sum_{i=1}^n w_i - n - \binom{k}{2}}$. Then the following relation holds.

Theorem 1.1 (Cai–Readdy) *The q -Stirling number of the second kind is given by*

$$S_q[n, k] = \sum_{w \in \mathcal{R}(n, k)} \text{wt}(w).$$

Readdy and I define a new pair of statistics $(A(\cdot), B(\cdot))$ on a subset of the RG -words, which we call *allowable RG -words*, such that $S_q[n, k]$ has a more compact expression on the subset. An RG -word $w \in \mathcal{R}(n, k)$ is *allowable* if every even entry appears exactly once. Denote by $\mathcal{A}(n, k)$ the set of all allowable RG -words in $\mathcal{R}(n, k)$. Let

$$A_i(w) = \begin{cases} w_i - 1 & \text{if } m_{i-1} \geq w_i, \\ 0 & \text{if } m_{i-1} < w_i \text{ or } i = 1, \end{cases} \quad \text{and} \quad B_i(w) = \begin{cases} 1 & \text{if } m_{i-1} > w_i, \\ 0 & \text{otherwise,} \end{cases}$$

where $m_i = \min(w_1, w_2, \dots, w_i)$. Define $A(w) = \sum_{i=1}^n A_i(w)$ and $B(w) = \sum_{i=1}^n B_i(w)$.

Theorem 1.2 (Cai–Readdy) *The q -Stirling numbers of the second kind can be expressed as a weighting over the set of allowable RG -words as follows:*

$$S_q[n, k] = \sum_{w \in \mathcal{A}(n, k)} q^{A(w)} \cdot (1 + q)^{B(w)}.$$

Let $a(n, k) = |\mathcal{A}(n, k)|$ be the cardinality of the set of allowable words. We call this the *allowable Stirling number of the second kind*. This is the new sequence **A256161** on *The On-line Encyclopedia of Integer Sequences* and it enjoys the following properties.

Proposition 1.3 (Cai–Readdy) *The allowable Stirling numbers of the second kind satisfy the following:*

- (i) $a(n, k) = a(n - 1, k - 1) + \lceil k/2 \rceil \cdot a(n - 1, k)$ for $n \geq 1$ and $1 \leq k \leq n$, with the boundary conditions $a(n, 0) = \delta_{n, 0}$.
- (ii) $a(n, 2) = n - 1$ and $a(n, n - 1) = \lfloor n/2 \rfloor \cdot \lceil n/2 \rceil$.

In order to understand the q -Stirling numbers more deeply, Readdy and I also give a poset structure on $\mathcal{R}(n, k)$, which we call *the Stirling poset of the second kind*, denoted by $\Pi(n, k)$, as follows. For $v, w \in \mathcal{R}(n, k)$ let $v = v_1 v_2 \cdots v_n \prec w$ if $w = v_1 v_2 \cdots (v_i + 1) \cdots v_n$ for some index i . The Stirling poset of the second kind is graded by the degree of the weight function wt . Thus the rank of the poset $\Pi(n, k)$ is $(n - k)(k - 1)$ and its rank generating function is given by $S_q[n, k]$.

By considering the repeated odd entries in an allowable RG -word, we find a Boolean algebra decomposition of Stirling poset of the second kind, which leads to a computation of the homology of the algebraic complex supported by this poset. See Figure 1 for an example of the Stirling poset $\Pi(5, 3)$ and its decomposition.

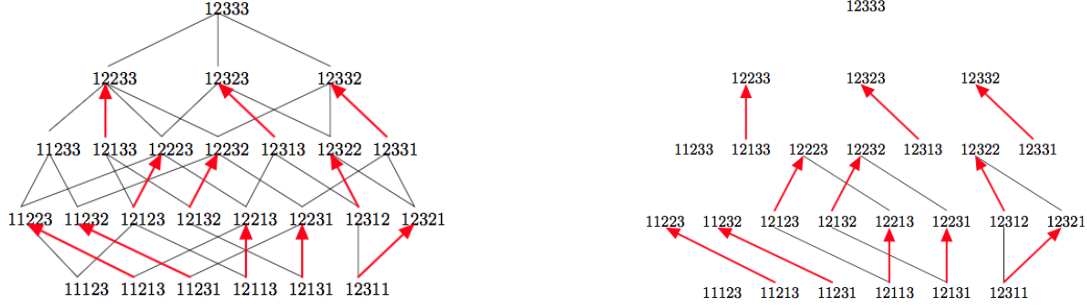


Figure 1: The Stirling poset $\Pi(5, 3)$ and its decomposition into Boolean algebras.

Theorem 1.4 (Cai–Readdy) *The Stirling poset of the second kind $\Pi(n, k)$ can be decomposed as the disjoint union of Boolean intervals*

$$\Pi(n, k) = \dot{\bigcup}_{w \in \mathcal{A}(n, k)} [w, \alpha(w)].$$

Furthermore, if an allowable word $w \in \mathcal{A}(n, k)$ has weight $\text{wt}'(w) = q^i \cdot (1 + q)^j$, then the rank of the element w is i and the interval $[w, \alpha(w)]$ is isomorphic to the Boolean algebra on j elements.

By defining a boundary map of the algebraic complex supported by the Stirling poset of the second kind $\Pi(n, k)$, and constructing an acyclic matching on the poset, we can apply Discrete Morse Theory to obtain a topological interpretation of Stembridge’s $q = -1$ phenomenon [23].

Theorem 1.5 (Cai–Readdy) *For the algebraic complex (\mathcal{C}, ∂) supported by the Stirling poset of the second kind $\Pi(n, k)$, a basis for the integer homology is given by the increasing allowable RG-words $\mathcal{A}(n, k)$. Furthermore,*

$$\sum_{i \geq 0} \dim H_i(\mathcal{C}, \partial; \mathbb{Z}) \cdot q^i = \left[\begin{matrix} n - 1 - \lfloor \frac{k}{2} \rfloor \\ \lfloor \frac{k-1}{2} \rfloor \end{matrix} \right]_{q^2}.$$

2 q -Stirling numbers of the first kind

The (unsigned) q -Stirling numbers of the first kind are defined by the recurrence formula

$$c_q[n, k] = c_q[n - 1, k - 1] + [n - 1]_q \cdot c_q[n - 1, k],$$

where $c_q[n, 0] = \delta_{n,0}$ and $[m]_q = 1 + q + \dots + q^{m-1}$. When $q = 1$, the Stirling number of the first kind $c(n, k)$ enumerates permutations in the symmetric group \mathfrak{S}_n having exactly k cycles. A combinatorial way to express q -Stirling numbers of the first kind is via rook placements; see de M edicis and Leroux [7].

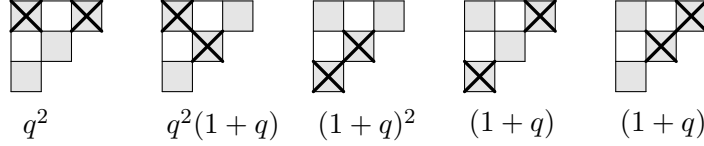


Figure 2: Computing the q -Stirling number of the first kind $c_q[4, 2]$ using $\mathcal{Q}(4, 2)$.

Theorem 2.1 (de Médicis–Leroux) *The q -Stirling number of the first kind is given by*

$$c_q[n, k] = \sum_{T \in \mathcal{P}(n, n-k)} q^{s(T)},$$

where the sum is over all rook placements of $n - k$ rooks on a staircase board of length n and $s(T)$ denotes the number of squares below the rooks in T .

As in the case of the q -Stirling number of the second kind, I do a similar analysis for the q -Stirling numbers of the first kind. I find a subset $\mathcal{Q}(n, n - k)$ of rook placements in $\mathcal{P}(n, n - k)$ so that the q -Stirling number of the first kind $c_q[n, k]$ has a more compact representation.

Theorem 2.2 (Cai–Readdy) *The q -Stirling number of the first kind is given by*

$$c_q[n, k] = \sum_{T \in \mathcal{Q}(n, n-k)} q^{s(T)} \cdot (1 + q)^{r(T)},$$

where $\mathcal{Q}(n, k)$ denotes rook placements in a shaded board of length n , $r(T)$ denotes the number of rooks not in the first row and $s(T)$ is as in Theorem 2.1.

Figure 2 shows an example of computing $c_q[4, 2]$ using shaded rook placements.

Let $d(n, k) = |\mathcal{Q}(n, n - k)|$. We call $d(n, k)$ the *allowable Stirling number of the first kind*. The following properties hold for $d(n, k)$.

Proposition 2.3 (Cai–Readdy) *The allowable Stirling numbers of the first kind $d(n, k)$ satisfy*

1. $d(n, k) = d(n - 1, k - 1) + \lceil (n - 1)/2 \rceil \cdot d(n - 1, k)$ with boundary conditions $d(n, 0) = \delta_{n,0}$, $d(n, n) = 1$ for $n \geq 0$ and $d(n, k) = 0$ when $k > n$.
2. Let $r(n) = \sum_{i=0}^k d(n, k)$ be the row sum of the allowable Stirling numbers of the first kind, then

$$d(n, 1) = \begin{cases} \left(\frac{n-1}{2}\right)!^2 & \text{for } n \text{ odd,} \\ \frac{n}{2} \cdot \left(\frac{n-1}{2}\right)!^2 & \text{for } n \text{ even,} \end{cases}$$

$$d(n, n - 1) = \left\lfloor \frac{n}{2} \right\rfloor \cdot \left\lceil \frac{n}{2} \right\rceil,$$

$$r(n) = d(n + 2, 1).$$

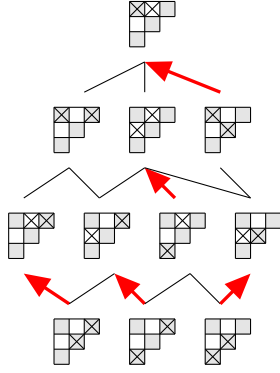


Figure 3: Example of $\Gamma(4, 2)$ with its matching. There is one unmatched rook placement in rank 2.

We define a poset structure on rook placements on a staircase shape board, namely, for rook placements T and T' in $\mathcal{P}(m, n)$, let $T \prec T'$ if T' can be obtained from T by either moving a rook to the left (west) or up (north) by one square. We call this poset *the Stirling poset of the first kind* and denote it by $\Gamma(m, n)$. It is straightforward to check that the poset $\Gamma(m, n)$ is graded of rank $(m - 1) + (m - 2) + \cdots + (m - n) = m \cdot n - \binom{n+1}{2}$ and its rank generating function is $c_q[m, m - n]$. Figure 3 gives an example of $\Gamma(4, 2)$.

We construct an acyclic matching on the Stirling poset of the first kind to give a Boolean algebra decomposition. Again we apply Discrete Morse Theory to compute the homology of the algebraic complex supported by this poset.

Theorem 2.4 (Cai–Readdy) *The Stirling poset of the first kind $\Gamma(n, k)$ can be decomposed as a disjoint union of Boolean intervals*

$$\Gamma(m, n) = \dot{\bigcup}_{T \in \mathcal{Q}(m, n)} [T, \alpha(T)].$$

For the algebraic complex (\mathcal{C}, ∂) supported by the Stirling poset of the first kind $\Gamma(n, k)$, a basis for the integer homology is given by the rook placements in $\mathcal{P}(m, n)$ having all of the rooks occur in shaded squares in the first row. Furthermore,

$$\sum_{i \geq 0} \dim(H_i(\mathcal{C}, \partial; \mathbb{Z})) \cdot q^i = q^{n(n-1)} \cdot \left[\begin{matrix} \lfloor \frac{m+1}{2} \rfloor \\ n \end{matrix} \right]_{q^2}.$$

3 Orthogonality of q -Stirling numbers

In [25] Viennot has some beautiful results in which he gave combinatorial bijections for orthogonal polynomials and their moment generating functions. One well-known relation between the ordinary signed Stirling numbers of the first kind and Stirling numbers of the second kind is their orthogonality. A bijective proof of the orthogonality of their q -analogues via 0-1 tableaux was given by de Mécidic and Leroux [7].

Letting $t = 1 + q$ we define the (q, t) -analogues of the Stirling numbers of the first and second kind. Readdy and I show that orthogonality holds combinatorially for the (q, t) -version of the Stirling numbers via a sign-reversing involution on ordered pairs of rook placements and RG -words.

Theorem 3.1 (Cai–Readdy) *The (q, t) -Stirling numbers are orthogonal, that is, for $m \leq n$*

$$\sum_{k=m}^n s_{q,t}[n, k] \cdot S_{q,t}[k, m] = \delta_{m,n} \quad \text{and} \quad \sum_{k=m}^n S_{q,t}[n, k] \cdot s_{q,t}[k, m] = \delta_{m,n},$$

where $s_{q,t}[n, k]$ is the signed version of the (q, t) -Stirling number of the first kind, that is, $s_{q,t}[n, k] = (-1)^{n-k} \cdot c_{q,t}[n, k]$. Furthermore, this orthogonality holds bijectively.

The results of Sections 1 through 3 appear in the article [2]. See [3] for an extended abstract.

4 01-permutations and major index

Let $\Omega(n, k) = \mathfrak{S}\{1^k, 0^{n-k}\}$ be the set of all 01-permutations consisting of k ones and $n - k$ zeros. For any $w = w_1 w_2 \cdots w_n \in \Omega(n, k)$, the *descent set* $D(w)$ of w is $D(w) = \{i : w_i > w_{i+1}\} \subseteq \{1, 2, \dots, n-1\}$. The *major index* of a 01-permutation w is defined to be the sum of all elements of $D(w)$:

$$\text{maj}(w) = \sum_{i \in D(w)} i.$$

The major index is closely related to the inversion numbers. In 1916 MacMahon [17, Page 315] showed that the two statistics $\text{inv}(\cdot)$ and $\text{maj}(\cdot)$ are equidistributed, that is,

$$\sum_{\pi \in \mathfrak{S}(0^{n-k}, 1^k)} q^{\text{maj}(\pi)} = \sum_{\pi \in \mathfrak{S}(0^{n-k}, 1^k)} q^{\text{inv}(\pi)} = \left[\begin{matrix} n \\ k \end{matrix} \right]_q,$$

where the Gaussian polynomial or the q -binomial $\left[\begin{matrix} n \\ k \end{matrix} \right]_q$ is given by $\left[\begin{matrix} n \\ k \end{matrix} \right]_q = \frac{[n]_q!}{[k]_q! [n-k]_q!}$. Foata [10] gave a bijection proof of this result in 1964.

Fu, Reiner, Stanton and Thiem [11] showed the q -binomial can be defined on a subset $\Delta(n, k)$ of 01-permutations in $\Omega(n, k)$ with a new pair of statistics $\text{inv}(\cdot)$ and $r(\cdot)$:

Theorem 4.1 (Fu–Reiner–Stanton–Thiem)

$$\sum_{w \in \Omega'(n, k)} q^{\text{inv}(w) - r(w)} \cdot (1 + q)^{r(w)} = \left[\begin{matrix} n \\ k \end{matrix} \right]_q.$$

I extend their results to the major index and give a combinatorial proof of the equidistribution of the statistics. I do this by describing an algorithm which determines a subset $\Omega'(n, k)$ of all 01-permutations in $\Omega(n, k)$. For a word $w \in \Omega'(n, k)$, I define a statistic $a(w)$ based on the algorithm. Then the pair of statistics $(\text{maj}(\cdot), a(\cdot))$ satisfy the following:

Theorem 4.2 (Cai) $\sum_{w \in \Omega'(n,k)} q^{\text{maj}(w)-a(w)} \cdot (1+q)^{a(w)} = \begin{bmatrix} n \\ k \end{bmatrix}_q$.

Furthermore, by constructing a bijective map $\sigma : \Delta(n, k) \rightarrow \Omega'(n, k)$, I show that the statistics $(\text{inv}(\cdot), r(\cdot))$ and $(\text{maj}(\cdot), a(\cdot))$ are equidistributed, that is,

Theorem 4.3 (Cai) *For any $\pi \in \Delta(n, k)$ and $\sigma(\pi) \in \Omega'(n, k)$, we have $\text{inv}(\pi) = \text{maj}(\sigma(\pi))$ and $r(\pi) = a(\sigma(\pi))$.*

This work appears in [1].

5 Application of RG -words

There are many q -identities involving q -Stirling numbers of the second kind. Most of them were proven by induction which gives little insight about the identities. It would be more revealing to give combinatorial proofs of the identities. Using RG -words we give a combinatorial proof of the following poset decomposition theorem.

Theorem 5.1 (Cai–Readdy) *The $(n-1)$ -fold Cartesian product of the m -chain has the decomposition*

$$(C_m)^{n-1} = \dot{\bigcup}_{1 \leq k \leq n} \dot{\bigcup}_{w \in \mathcal{A}(n,k)} [w, \alpha(w)] \times C_{m-1} \times C_{m-2} \times \cdots \times C_{m-k+1}, \quad (5.1)$$

where C_j denotes the chain on j elements, and $[w, \alpha(w)]$ is a Boolean interval.

By considering the rank generating function of identity (5.1), we obtain a poset theoretic proof of Carlitz's identity.

Corollary 5.2 (Cai–Readdy) *There is a poset theoretic proof of Carlitz's identity:*

$$[m]_q^n = \sum_{k=0}^n q^{\binom{k}{2}} \cdot S_q[n, k] \cdot [k]! \cdot \begin{bmatrix} m \\ k \end{bmatrix}_q.$$

We prove other q -Stirling identities combinatorially via RG -words. This includes the generating function of q -Stirling numbers of the second kind by Gould [13], the q -Vandermonde convolution for Stirling numbers due to Chen [6], de Médicis and Leroux [8] and the q -Frobenius identity by Garsia and Remmel [12]. See my paper [4] for more details.

References

- [1] Y. CAI, A q -($1 + q$) analogue of the major index, in preparation.
- [2] Y. CAI AND M. READDY, q -Stirling numbers: A new view, submitted to *J. Combin. Theory Ser. A*, arXiv: 1506.03249, 30 pages.
- [3] Y. CAI AND M. READDY, Negative q -Stirling numbers, *Proceedings of the 27th International Conference on Formal Power Series and Algebraic Combinatorics* (2015), 583–594.
- [4] Y. CAI AND M. READDY, q -Stirling identities revisited, preprint 2015, 12 pages.
- [5] L. CARLITZ, On abelian fields, *Trans. Amer. Math. Soc.* **35** (1933), 122–136.
- [6] W. Y. C. CHEN, Context-free grammars, differential operators and formal power series, *Theoret. Comput. Sci.* **117** (1993), 113–129.
- [7] A. DE MÉDICIS AND P. LEROUX, A unified combinatorial approach for q - (and p, q -) Stirling numbers, *J. Statist. Plann. Inference* **34** (1993), 89–105.
- [8] A. DE MÉDICIS AND P. LEROUX, Generalized Stirling numbers, convolution formulae and p, q -analogues, *Canad. J. Math.* **47** (1995), 474–499.
- [9] R. EHRENBORG AND M. READDY, Juggling and applications to q -analogues, *Discrete Math.* **157** (1996), 107–125.
- [10] D. FOATA, Sur un énoncé de MacMahon, *C. R. Acad. Sci. Paris* **258** (1964), 1672–1675.
- [11] S. FU, V. REINER, D. STANTON AND N. THIEM, The negative q -binomial, *Electron. J. Combin* **19** (2012), Paper P36, 24 pp.
- [12] A. M. GARSIA AND J. B. REMMEL, Q -counting rook configurations and a formula of Frobenius, *J. Combin. Theory Ser. A* **41** (1986), 246–275.
- [13] H. W. GOULD, The q -Stirling numbers of the first and second kinds, *Duke Math. J.* **28** (1961), 281–289.
- [14] M. JÖLLENBECK AND V. WELKER, Minimal resolutions via algebraic discrete Morse theory, *Mem. Amer. Math. Soc.* **197** (2009, no. 923), 74 pp.
- [15] D. KOZLOV, “Combinatorial Algebraic Topology,” Algorithms and Computation in Mathematics (Volume. 21), Springer, 2008.
- [16] P. LEROUX, Reduced matrices and q -log-concavity properties of q -Stirling Numbers, *J. Combin. Theory Ser. A* **54** (1990), 64–84.
- [17] P. A. MACMAHON, Two applications of general theorems in combinatory analysis: (1) to the theory of inversions of permutations; (2) to the ascertainment of the numbers of terms in the development of a determinant which has amongst its elements an arbitrary number of zeros, *Proc. London. Math. Soc.* (2) **15** (1916), 314–321.
- [18] S. MILNE, A q -analog of restricted growth functions, Dobinski’s equality, and Charlier polynomials, *Trans. Amer. Math. Soc.* **245** (1978), 89–118.
- [19] V. REINER, D. STANTON AND D. WHITE, The cyclic sieving phenomenon, *J. Combin. Theory Ser. A* **108** (2004), 17–50.
- [20] G.-C. ROTA, The number of partitions of a set, *Amer. Math. Monthly* **71** (1964), 498–504.
- [21] E. SKÖLDBERG, Morse theory from an algebraic viewpoint, *Trans. Amer. Math. Soc.* **358** (2006), 115–129.
- [22] R. P. STANLEY, “Enumerative Combinatorics, Vol. I, 2nd edition,” Cambridge Studies in Advanced Mathematics (No. 49), Cambridge University Press, 2012.
- [23] J. STEMBRIDGE, Some hidden relations involving the ten symmetry classes of plane partitions, *J. Combin. Theory Ser. A* **68** (1994), 372–409.

- [24] J. STEMBRIDGE, Canonical bases and self-evacuating tableaux, *Duke Math. J.* **82** (1996), 585–606.
- [25] G. VIENNOT, A combinatorial theory for general orthogonal polynomials with extensions and applications, in *Polynômes Orthogonaux et Applications* (Bar-le-Duc, 1984), 139–157, *Lecture Notes in Mathematics Volume 1171*, Springer, Berlin, 1985.
- [26] M. WACHS AND D. WHITE, p, q - Stirling numbers and set partition statistics, *J. Combin. Theory Ser. A* **56** (1991), 27–46.
- [27] D. WHITE, Interpolating set partition statistics, *J. Combin. Theory Ser. A* **68** (1994), 262–295.