RESEARCH STATEMENT

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This research statement is divided into three sections. First we discuss Ehrhart theory for lattice polytopes, some open questions, and results in these directions. After this, we look at the generalization of the theory to rational polytopes, see how such polytopes arise naturally, and note how problems increase in complexity while still remaining tractable. Lastly we discuss how, even though the techniques needed are often highly technical and advanced, there are plenty of questions that are accessible to those with undergraduate mathematics backgrounds. Some questions are presented as examples.

1. Ehrhart Theory and Reflexive Polytopes

Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a lattice polytope of dimension d, that is, the convex hull of a finite number of vectors in $\mathbb{Z}^n$ whose affine span has dimension d. Consider the counting function $\mathcal{L}_\mathcal{P}(m) = |m\mathcal{P} \cap \mathbb{Z}^n|$, where $m\mathcal{P}$ is the m-th dilate of $\mathcal{P}$. The *Ehrhart series* of $\mathcal{P}$ is

$$E_\mathcal{P}(t) := 1 + \sum_{m \in \mathbb{Z}_{\geq 1}} \mathcal{L}_\mathcal{P}(m) t^m .$$

Combining two well-known theorems due to Ehrhart [13] and Stanley [28], there exist values $h_0^*, \ldots, h_d^* \in \mathbb{Z}_{\geq 0}$ with $h_0^* = 1$ such that

$$E_\mathcal{P}(t) = \frac{\sum_{j=0}^d h_j^* t^j}{(1-t)^{d+1}} .$$

We say the polynomial $h_\mathcal{P}^*(t) := \sum_{j=0}^d h_j^* t^j$ is the *h^*-polynomial* of $\mathcal{P}$ (sometimes referred to as the δ-polynomial of $\mathcal{P}$) and the vector of coefficients $h^*(\mathcal{P})$ is the *h^*-vector* of $\mathcal{P}$. That $E_\mathcal{P}(t)$ is of this rational form is equivalent to $|m\mathcal{P} \cap \mathbb{Z}^n|$ being a polynomial in m of degree at most d; the non-negativity of the h^*-vector is an even stronger property.

Recent work has focused on determining when $h^*(\mathcal{P})$ is unimodal, that is, when there exists some k for which $h_0^* \leq \cdots \leq h_k^* \geq \cdots \geq h_d^*$. The specific sequence in question may not be of particular interest, but unimodal behavior may suggest an underlying structure of $\mathcal{P}$ that is not immediately apparent. Thus, the proofs of various h^*-vectors being unimodal are often more enlightening than the sequences themselves. There are a number of approaches possible for proving unimodality, taken from fields such as Lie theory, probability, and others [29], but it is generally very difficult to prove.

Even for highly structured polytopes, unimodality can be difficult to show. Such a class is the following.

Definition 1.1. A lattice polytope $\mathcal{P}$ is called *reflexive* if $0 \in \mathcal{P}^\circ$ and its *(polar) dual*

$$\mathcal{P}^\Delta := \{y \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for all } x \in \mathcal{P}\}$$

is also a lattice polytope.

Reflexive polytopes have been the subject of a large amount of recent research [3, 4, 7, 8, 15, 18, 22, 24]. It is known from work of Lagarias and Ziegler [20] that there are only finitely many reflexive polytopes (up to unimodular equivalence) in each dimension, with one reflexive in dimension one, 16 in dimension two, 4 319 in dimension three, and 473 800 776 in dimension four according to computations by Kreuzer and Skarke [19]. The number of five-and-higher-dimensional reflexives is unknown. One of the reasons reflexives are of interest is the following.

Theorem 1.2 (Hibi, [18])**.** A d-dimensional lattice polytope $\mathcal{P} \subset \mathbb{R}^d$ containing the origin in its interior is reflexive if and only if $h^*(\mathcal{P})$ satisfies $h^*_i = h^*_{d-i}$.

Hibi [17] conjectured that every reflexive polytope has a unimodal h^*-vector. Counterexamples to this were found in dimensions 6 and higher by Mustaţă and Payne [22, 24]. However, Hibi and Ohsugi [23] also asked whether or not every normal reflexive polytope has a unimodal h^*-vector; a related notion is that of an integrally closed polytope.

Definition 1.3. A lattice polytope $\mathcal{P} \subseteq \mathbb{R}^n$ is *integrally closed* if, for every $x \in m\mathcal{P} \cap \mathbb{Z}^n$, there exist $x_1, \ldots, x_m \in \mathcal{P} \cap \mathbb{Z}^n$ such that $x = x_1 + \cdots + x_m$.

While the terms integrally closed and normal are often used interchangeably, these are not synonymous [14]. A first step to understanding unimodality with all of these restrictions is to first ask about unimodality in the simplicial case. This leads to the following question.

Question 1.4. Is the h^*-vector of an integrally closed, reflexive simplex unimodal?

A priori there is no indication that this *should* be true. Fortunately there is a classification algorithm for reflexive simplices [11] that associates a *type* (Q, λ) to each, which might be extremely helpful for answering Question 1.4. Since there is still a large number of simplices to work through, we would like to avoid redundancy when searching through them. Braun and I investigated a method of constructing polytopes that, with appropriate restrictions, preserves properties that one may be interested in.

Construction 1.5. Suppose $\mathcal{P} \subseteq \mathbb{R}^n$ and $\mathcal{Q} \subseteq \mathbb{R}^m$ are full-dimensional polytopes with $0 \in \mathcal{P}$ and $\{v_0, \ldots, v_k\}$ denoting the vertices of $\mathcal{Q}$. Then for each $i = 0, 1, \ldots, k$ there is a polytope $\mathcal{P} *_i \mathcal{Q}$ defined by

$$\mathcal{P} *_i \mathcal{Q} := \operatorname{conv}\{(\mathcal{P} \times 0^m) \cup (0^n \times \mathcal{Q} - v_i)\} \subseteq \mathbb{R}^{n+m}.$$

Given the types of $\mathcal{P}$ and $\mathcal{Q}$, computing the type of $\mathcal{P} *_i \mathcal{Q}$ is simple. As we will see, this ability to compute types of reflexive simplices is helpful in directing approaches to Question 1.4.

Theorem 1.6 (Braun and Davis, 2014)**.** If $\mathcal{P} = \operatorname{conv}\{v_0, \ldots, v_n\} \subseteq \mathbb{R}^n$ and $\mathcal{Q} = \operatorname{conv}\{w_0, \ldots, w_m\} \subseteq \mathbb{R}^m$ are full-dimensional reflexive simplices of types $((p_0, \ldots, p_n), \lambda)$ and $((q_0, \ldots, q_m), \mu)$, respectively, then $\mathcal{P} *_i \mathcal{Q}$ is a reflexive simplex of type

$$\left(\frac{1}{d}(q_i p_0, q_i p_1, \ldots, q_i p_n, s q_0, s q_1, \ldots, \widehat{s q_i}, \ldots, s q_m), d\right),$$

where $s = \sum_{j=0}^n p_j$ and $d = \gcd(q_i, \sum_{j=0}^n p_j)$.

Of course, the construction itself can be done with any two polytopes, as long as $0 \in \mathcal{P}$. We can get more some mileage out of the construction in the following ways.

Theorem 1.7 (see [9])**.** Suppose $\mathcal{P} \subseteq \mathbb{R}^n$ and $\mathcal{Q} \subseteq \mathbb{R}^m$ are full-dimensional polytopes with $0 \in \mathcal{P}$ and $\{v_0, \ldots, v_k\}$ denoting the vertices of $\mathcal{Q}$. Then for each $i = 0, 1, \ldots, k$ the polytope formed by

$$\mathcal{P} *_i \mathcal{Q} := \mathrm{conv}\{(\mathcal{P} \times 0^m) \cup (0^n \times \mathcal{Q} - v_i)\} \subseteq \mathbb{R}^{n+m}$$

is a free sum. Moreover, if $0 \in \mathcal{P}^\circ$ and $\mathcal{P}$ and $\mathcal{Q}$ are both reflexive, then $\mathcal{P} *_i \mathcal{Q}$ is also reflexive.

Theorem 1.8 (Braun and Davis, 2014)**.** If $\mathcal{P}$ and $\mathcal{Q}$ are any integrally closed polytopes with $0 \in \mathcal{P}^\circ$ and $\mathcal{P}$ reflexive, then $\mathcal{P} *_i \mathcal{Q}$ is integrally closed.

These theorems together provide a way to produce new h^*-vectors from old.

Theorem 1.9 (see [5, 9])**.** Let $\mathcal{P} \subseteq \mathbb{R}^n$ and $\mathcal{Q} \subseteq \mathbb{R}^m$ be lattice polytopes with $0 \in \mathcal{P}^\circ$. Then

(1) their h^*-polynomials multiply, i.e.

$$h^*_{\mathcal{P} *_i \mathcal{Q}}(t) = h^*_{\mathcal{P}}(t) h^*_{\mathcal{Q}}(t),$$

(2) if $\mathcal{P}$ and $\mathcal{Q}$ are both simplices, then so is $\mathcal{P} *_i \mathcal{Q}$, and

(3) if $\mathcal{P}$ and $\mathcal{Q}$ are both integrally closed and reflexive, then so is $\mathcal{P} *_i \mathcal{Q}$.

So, in the context of Question 1.4, if $h^*(\mathcal{P})$ and $h^*(\mathcal{Q})$ are unimodal, then so is $h^*(\mathcal{P} *_i \mathcal{Q})$. Thus if we want to search for a counterexample to Question 1.4, we want to rule out any that decompose as two smaller-dimensional simplices satisfying the conditions of the previous theorem. Although this helps to reduce the number of simplices to sort through, the question is still wide open. An approach using representation theory can be useful to Ehrhart theory problems in this area, adapting methods used in [25].

Theorem 1.10 (see [25])**.** If a reflexive simplex $\mathcal{P}$ "carries" a representation of $\mathrm{sl}_2(\mathbb{C})$, then $h^*(\mathcal{P})$ is unimodal.

This perspective is rather unexplored, which leaves a lot of questions to ask.

Question 1.11. Which reflexive simplices carry a representation of $\mathrm{sl}_2(\mathbb{C})$? Can nontrivial classes of reflexive simplices be constructed that either do or do not carry such a representation? How can this be related to the type of a reflexive simplex?

Determining whether a representation is carried is equivalent to determining the existence of linear operators satisfying certain equalities. Doing this in general can be challenging, but examples in lower dimensions may give hints to any larger structure, since the number of lattice points involved can be kept low. I will start by looking at these lower-dimensional simplices to collect data, and I will examine how different constructions of free sums affect the existence of a representation of $\mathrm{sl}_2(\mathbb{C})$. Another approach comes from commutative algebra: for a lattice simplex $\mathcal{P} \subseteq \mathbb{R}^n$ with vertices $\{v_0, \ldots, v_n\}$, let

$$\Pi(\mathcal{P}) := \left\{ \sum_{i=0}^{n} c_i v_i \middle| 0 \leq c_i < 1 \right\}.$$

The set of lattice points in $\Pi(\mathcal{P})$ correspond to the zero-dimensional graded algebra

$$k[\Pi(\mathcal{P})] := k[x^a z^m \in \mathbb{R}^{n+1} | x \in m\mathcal{P} \cap \mathbb{Z}^n]/(x^{v_0}z, \dots, x^{v_n}z),$$

where $x^a := x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$. Through the work of Harima et al. [16], one can show that the h^*-vector of $\mathcal{P}$ is unimodal if and only if $k[\Pi(\mathcal{P})]$ has a weak Lefschetz element. Experimental data suggests that a weak Lefschetz element can be found much of the time, however they do not necessarily have to exist [9]. What are also not clear are the conditions necessary or sufficient for a weak Lefschetz element to exist.

Question 1.12. For which reflexive simplices $\mathcal{P}$ do $k[\Pi(\mathcal{P})]$ have a weak Lefschetz element? What do the weak Lefschetz elements look like? Is there a way to relate weak Lefschetz elements induced from two polytopes in a free sum?

Again, low-dimensional examples will be helpful here in experimenting with whether or not weak Lefschetz elements exist. There may be multiple such elements, in which case it would be helpful to determine when a polytope $\mathcal{P}$ induces exactly one weak Lefschetz element of a particular form, say, by taking the sum of all degree-one elements of $\Pi(\mathcal{P})$. Identifying polytopes that induce a single weak Lefschetz element may make it easier to predict the form of an element when taking free sums. This is another approach with a lot of room for exploration. For this work, I will expand upon computer code that I have already written for identifying whether or not a particular reflexive simplex has a weak Lefschetz element. This will make approaching the problem much more efficient.

2. Rational Ehrhart Theory

It is natural to also ask about lattice points contained in scalings of polytopes whose vertices are in $\mathbb{Q}^n$ rather than strictly $\mathbb{Z}^n$. In this case, the function $\mathcal{L}_{\mathcal{P}}(m)$ is a *quasipolynomial*, and there are variations in the rational form of $E_{\mathcal{P}}(t)$. Rational polytopes arise in many settings, one of which is a variation of the well-known Birkhoff polytope.

Definition 2.1. The *Birkhoff polytope* is the set of $n \times n$ matrices with real nonnegative entries such that each row and column sum is 1.

We denote this polytope by B_n and note that it is also often referred to as the polytope of real $n \times n$ doubly-stochastic matrices or the polytope of $n \times n$ magic squares. The fact that B_n is a polytope is due to the Birkhoff-von Neumann theorem, which finds that B_n is the convex hull of the permutation matrices. The h^*-vector of the Birkhoff polytope is difficult to compute in general, and is known only for $n \leq 9$; its volume only for $n \leq 10$ [6]. As limited as the data is, it has still been shown that $h^*(B_n)$ is symmetric as well as unimodal [2, 26, 27].

On the other hand, little is known about the polytope Σ_n obtained by intersecting B_n with the hyperplanes $x_{ij} = x_{ji}$ for all i, j, that is, by requiring the matrices in B_n to be symmetric. Nothing is new when $n \leq 2$, but complications arise once $n \geq 3$ since the vertices of Σ_n are no longer always integral. They are contained in the set

$$L_n = \left\{ \frac{1}{2}(P + P^T) | P \in \mathbb{R}^{n \times n} \text{ is a permutation matrix} \right\},$$

but L_n is not necessarily equal to the vertices of Σ_n. A description of the vertices and a generating function for the number of them can be found in [31]. The h^*-vector of Σ_n is known to be symmetric [30] and some values of $E_{\Sigma_n}(t)$ have been computed in a reduced form for small n [32], but it is still unknown whether the h^*-vector is always unimodal in this case.

Since there are so many more tools available for lattice polytopes, one strategy is to consider a related lattice polytope and see what information is retained about the rational one. One way to do this is to scale Σ_n by two, resulting in a polytope that is combinatorially equivalent to Σ_n but with integral vertices. We denote this dilation by S_n. Starting from this simple alteration I have been able to show the following.

Theorem 2.2 (see [12])**.** Let Σ_n and S_n be as above. If $n = 2k$ for some $k \in \mathbb{Z}_{>0}$, then

(1) $h^*(\Sigma_n) = (h_0^*, h_1^*, \ldots, h_{2d}^*)$ for $d = 2k^2 - 2k + 1$, and
(2) $h^*(S_n) = (h_0^*, h_2^*, \ldots, h_{2d-2}^*, h_{2d}^*)$ is unimodal.

This partial progress leads to a few natural questions:

Question 2.3.

(1) When n is even, how can we approach the odd-index terms for $h^*(\Sigma_n)$?
(2) In general, how do we prove unimodality of an h^*-vector without also having symmetry?

As shown in [12], each lattice point of S_n has a corresponding pseudograph. Theorem 2.2 is related to the integral closure of S_n, which can be interpreted as a certain way of decomposing the corresponding graphs. However, Σ_n is not integrally closed. The odd-index terms may be related to the "irreducible" pseuographs, in which case these are worth examining. Very little is known about such graphs, so I plan to continue this work by asking how many there are for fixed n. I expect these pseudographs to have certain structure that is reflected as the odd-index entries. This may also give some insight into the behavior of h^*-vectors for odd n, when $h^*(S_n)$ is not symmetric. The proof of Theorem 2.2 relies on showing that a (regular) unimodular triangulation of S_n exists. If the triangulation is explicitly known, then the Ehrhart series can be recovered through knowing the Ehrhart series of simplices and using inclusion-exclusion on the elements of the triangulation. Unfortunately, concrete descriptions of the triangulations are not entirely clear, which leaves the unimodality question open. A more general question is the following.

Question 2.4. Given an h^*-vector $(h_0^*, \ldots, h_d^*)$, what is the combinatorial interpretation (if any) of h_i^*?

It has been shown for any lattice polytope $\mathcal{P} \subseteq \mathbb{R}^n$ that $h_0^* = 1$, $h_1^* = |\mathcal{P} \cap \mathbb{Z}^n| - \dim \mathcal{P} - 1$, and if h_s^* is the highest-index nonzero entry of $h^*(\mathcal{P})$, then s is the smallest integer factor for which $s\mathcal{P}$ contains an interior lattice point and $\dim \mathcal{P} - h_s^* + 1$ is the number of lattice points in the interior of $s\mathcal{P}$. There are additional results for other special cases, but the question is likely unanswerable in full generality.

The idea of forming new polytopes from the lattice points of a rational polytope was also mentioned in a discussion of Gelfand-Tsetlin polytopes by Alexandersson [1]. Gelfand-Tsetlin polytopes are defined from arrays of real numbers satisfying certain inequalities, and are not necessarily lattice polytopes. Nonetheless, their Ehrhart counting functions can still be true polynomials instead of quasipolynomials [21]. Aside from their combinatorial interest, Gelfand-Tsetlin polytopes are important for their connections to representations of $\mathrm{gl}_n\mathbb{C}$. Little is known about the Ehrhart theory of Gelfand-Tsetlin polytopes, as evidenced by the multiple open questions mentioned in [1], one of which is the following.

Question 2.5. For each degree d, is there a finite number of Ehrhart polynomials of degree d that can be obtained as an Ehrhart polynomial of some Gelfand-Tsetlin polytope?

For this question, I will look at simple Gelfand-Tsetlin polytopes and determine which lattice polytopes have the same Ehrhart polynomials. Lattice polytopes are understood much better, so I will work to identify an infinite class of lattice polytopes whose h^*-vectors coincide with a particular infinite class of Gelfand-Testlin polytopes. I will then experiment with the lattice polytopes to examine any common behaviors with the Gelfand-Tsetlin polytopes.

3. Research Mentoring

There is an abundance of work that can be done from an Ehrhart theoretic perspective that does not necessarily require an extensive knowledge of the field. For example, two of my coauthors in [10] were undergraduates at the time while I had only taken a single combinatorics course in my life up to that point. The questions we examined were accessible to students with a fairly limited combinatorics background, but answering them was still challenging. In my future work I intend to continue such a tradition of exposing students to research-level mathematics at an early stage.

There are many ways to connect Ehrhart theory with other areas of combinatorics: polytopes arise in different ways using graphs, partially ordered sets, compositions and partitions, and others. These are objects that are easily accessible to undergraduates, and questions are easy to produce. For example, the edge polytope of a graph does not require knowing anything about graphs beyond the most basic definitions. From here, we could restrict our attention to certain classes of graphs and look for common properties of their edge polytopes; in the other direction, we could ask what graphs are produced if we require an edge polytope to have additional structure.

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