My research lies within the intersection of algebraic and tropical geometry. Namely, I am interested in using combinatorial techniques to classify all Mori dream spaces among a class of spaces known as projectivized toric vector bundles, here denoted as $\mathbb{P}\mathcal{E}$. This research statement details my contributions to these efforts and proposes possible directions for future research. It concludes by discussing my involvement in a long-term, ongoing undergraduate research project.

1 Mori Dream Spaces

In much of algebraic geometry, the fundamental objects of study are varieties. Recall that, in $\mathbb{P}^n_k$, a *projective variety* is the zero-locus of some finite family of homogeneous polynomials from $k[x_0, ..., x_n]$. A variety X is called *normal* if, for every point $p \in X$, the stalk $\mathcal{O}_{X,p}$ of the sheaf of rational functions on X is an integrally closed domain

Definition 1.1. For a normal, projective variety X with finitely generated class group, the *Cox ring* of X, denoted $\text{Cox}(X)$ or $\mathcal{R}(X)$, is defined as

$$\mathcal{R}(X) = \bigoplus_{[D] \in \mathcal{CL}(X)} H^0(X, [D])$$

where $H^0(X, D) = \{f \in K(X) \mid (\text{Div}(f) + D) \text{ is effective}\}$ is the group of global sections of D on X.

We note that the sum is taken over divisor classes and is independent of the choice of representative as, for any two divisors D and D' which differ by a principal generator, i.e. are elements of the same class, we have $H^0(X, D) \cong H^0(X, D')$.

While there are many definitions of Mori dream spaces, the following by Hu and Keel [HK00] is common in the literature and is the one used throughout this manuscript.

Definition 1.2. A *Mori dream space* X is a normal, projective variety (over $\mathbb{C}$) for which the Cox ring $R(X)$ is finitely generated.

Projective spaces and, more generally, toric varieties and flag varieties are all well-known examples of Mori dream spaces. However, not all "nice" operations on "nice" spaces produce Mori dream spaces. For example, in [CT07], Castravet and Tevelev considered blowing up r points on $\mathbb{P}^n$. They found that this operation gives a Mori dream space if and only if the following inequality holds:

$$\frac{1}{n+1} - \frac{1}{(n+1)-r} > \frac{1}{2}.$$

2 (Projectivized) Toric Vector Bundles

For my research, I considered a fan Σ and vector bundles over $Y(\Sigma)$, the toric variety associated to the fan Σ. A *toric vector bundle*, denoted $\mathcal{E}$, is such a vector bundle over $Y(\Sigma)$ along with the information of a torus action on $\mathcal{E}$ which is linear on the fibers and for which the projection map π is equivariant, i.e. the following diagram commutes when the vertical maps are the projection π and the horizontal maps are the action of the torus $\mathbb{T}$.

$$\begin{array}{ccc} \mathbb{T} \times E & \longrightarrow & E \\ \downarrow & & \downarrow \\ \mathbb{T} \times Y(\Sigma) & \longrightarrow & Y(\Sigma) \end{array}$$

We then obtain a *projectivized toric vector bundle* from a toric vector bundle by replacing each fiber with its respective projective space. Projectivized toric vector bundles exhibit some nice behavior. For example, the Picard group of a projectivized toric vector bundle $\mathbb{P}\mathcal{E}$, denoted $\mathrm{Pic}(\mathbb{P}\mathcal{E})$ can be decomposed as

$$\mathrm{Pic}(\mathbb{P}\mathcal{E}) \cong \mathrm{Pic}(Y(\Sigma)) \times \mathbb{Z}$$

where the line bundles on $\mathbb{P}\mathcal{E}$ are obtained from pullbacks from $Y(\Sigma)$. We also have that projectivized toric vector bundles are smooth and projective over the base field. They are more general than toric vector bundles, but still carry an essentially combinatorial description [Kly89], [KMa]. This led Hering, Payne, and Mustaţă to ask when a projectivized toric vector bundle $\mathbb{P}\mathcal{E}$ is a Mori dream space [HMP10]. The following two theorems describe some of the first major results along these lines.

Theorem 2.1 (Hausen, Süß, [HS10])**.** *Projectivized tangent bundles of toric varieties are Mori dream spaces.*

Theorem 2.2 (González, [Gon12])**.** *For a rank 2 toric vector bundle $\mathcal{E}$, the projectivized toric vector bundle $\mathbb{P}\mathcal{E}$ is a Mori dream space.*

These results serve as the foundation for my continued exploration into Mori dream spaces, approaching the following questions. In what follows, we say that a bundle $\mathcal{E}$ is a *Mori dream space bundle* if $\mathbb{P}\mathcal{E}$ is a Mori dream space.

1. *When is a rank r projectivized toric vector bundle a Mori dream space?*

2. *For $\mathcal{E}$ a Mori dream space bundle, when is $\mathcal{E} \oplus \mathcal{E}$ and, more generally, $\mathcal{E} \otimes V$ a Mori dream space bundle for V a finitely generated vector space?*

The first question, regarding rank 3 projectivized toric vector bundles, has already been explored some in the literature. There are known examples of such spaces which are also Mori dream as well as rank 3 projectivized toric vector bundles which are not Mori dream spaces. However, no complete classification currently exists. In Section 3, we describe a family of rank 3 projectivized toric vector bundles which are all Mori dream spaces. To do so, we find it beneficial to pass to tropical geometry to obtain a combinatorial description of the problem. We briefly review the necessary definitions from tropical geometry.

Definition 2.3. Let L be a linear ideal with tropicalized linear space

$$\mathrm{Trop}(L) = \bigcap_{f \in L} V\left(\mathrm{trop}(f)\right).$$

Then, for τ a maximal face of the Gröbner fan of L, the *apartment associated to* τ, denoted A_τ, is the intersection of τ with the tropicalized linear space of L, i.e.

$$A_\tau = \tau \cap \mathrm{Trop}(L).$$

Definition 2.4. For a fan Σ and linear ideal L, a *diagram* D of (Σ, L) is a matrix whose rows are indexed by the rays of Σ satisfying:

1. each row of D is a point in $\text{Trop}(L)$,
2. if $p_{i_1}, \ldots, p_{i_l} \in \Sigma(1)$ are rays contained in the same face, then the corresponding rows live in a common apartment of $\text{Trop}(L)$.

As described in [KM19], toric vector bundles can be parameterized by the data (L, D). Therefore, we seek to explore properties of toric vectur bundles as a function of this data. In particular, our work considers the case in which L is a generic linear ideal. When this happens, the associated toric vector bundle is called *uniform*.

3 Rank r Projectivized Toric Vector Bundles

Question 3.1. *When is a rank r projectivized toric vector bundle a Mori dream space?*

The following theorem gives a sufficient condition for rank 3 bundles to be a Mori dream space, which is then generalized in Theorem 3.3.

Theorem 3.2 (G, Kaveh, Manon)**.** *A rank 3 uniform projectivized toric vector bundle $\mathbb{P}\mathcal{E}$ is a Mori dream space if there exists a subdivision of the Gröbner fan so that every pair of points associated to the diagram D lives in a common apartment iff every pair of rows of D has two common zero columns.*

The common apartment condition for a Mori dream space, previously referred to in [KMa] as the vector bundle being *formal*, has now been given the more descriptive name of a *complete intersection bundle*. Expanding on the work of Kaveh and Manon, the following condition given a sufficient MDS condition for a uniform rank r bundle.

Theorem 3.3. *Let D be the $n \times m$ diagram corresponding to a rank r uniform projectivized toric vector bundle. Then, for every choice of p rows, if all of the nonzero entries appear in $r + p - 2$ columns, the bundle is a Mori dream space bundle.*

Example 3.4. Let $L = \langle \sum_{i=1}^{6} y_i, \sum_{i=1}^{6} i y_i \rangle \subset \mathbb{C}[y_1, .., y_6]$. Considering

$$D = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

we can check every subset of 1, 2, and 3 rows to confirm that we always have the nonzero elements of those rows contained in $3 + p - 2 = p + 1$ columns. So the bundle corresponding to this L and D is a Mori dream space bundle. However, if we change the last row slightly to make

$$D' = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

we now have that the nonzero elements of first and third rows are contained in 4 columns and $4 > 3 + 2 - 2 = 3$, violating the sufficient condition. However, despite not being a complete intersection bundle, algorithmic calculation of the Cox ring can be done to show that this bundle is still a Mori dream space (See Section 4).

Expanding on Example 3.4, my future work will include trying to classify rank r projectivized toric vector bundles that are not complete intersection bundles but still Mori dream spaces. In particular, I am interested in exploring the cases where the non-zero entries of D are entries ≥ 2, which, in experimentation, have already shown to have more complex behavior than a $(0,1)-D$.

4 Sums of Bundles

Question 4.1. *If $\mathbb{P}\mathcal{E}$ is a Mori dream space and V is a finite-dimensional vector space, when is $\mathbb{P}(\mathcal{E} \otimes V)$ a Mori dream space?*

We find that the answer to this question is closely related to the study of flag varieties. Therefore, before stating the main results, we begin with a review of some key definitions.

Let E be a vector space of dimension r. Given a strictly increasing sequence of integers $I = \{i_1, \ldots, i_d\} \subseteq \{1, ..., r-1\}$, the *flag variety* $\mathcal{FL}_I(E)$ is the space of flags $0 \subset V_1 \subset \ldots \subset V_d \subset E$ where each V_j is a subspace of dimension i_j. A bundle is called a *full flag bundle* when $I = \{1, ..., r-1\}$. We note that bundles of flag varieties are natural generalizations of projectivized vector bundles. For my work, we focused on the case of flag bundles over toric varieties. A *toric flag bundle* over $X(\Sigma)$ is a bundle with fibers isomorphic to $\mathcal{FL}_I(E)$, equipped with a compatible torus action. Therefore, we ask: *When is $\mathcal{FL}_I(\mathcal{E})$ a Mori dream space?* The answer to this is linked to Question 3.1 by the following theorem.

Theorem 4.2 (G, Manon)**.** *Let $dim(V) = l$. Then the projective bundle $\mathbb{P}(\mathcal{E} \otimes V)$ is a Mori dream space if and only if the bundle $\mathcal{FL}_I(\mathcal{E})$ is a Mori dream space for all $|I| \leq l$. In particular, for a toric vector bundle $\mathcal{E}$, the full flag bundle $\mathcal{FL}(\mathcal{E})$ is a Mori dream space if and only if $\mathbb{P}(\mathcal{E} \otimes V)$ is a Mori dream space for all finite dimensional vector spaces V.*

For an example, we consider *sparse uniform bundles* [KMb], where L is a generic linear ideal and D, up to certain equivalences, has at most one non-zero, positive entry per row. Within these bundles, we have an extremal family: sparse hypersurface bundles. In the context of this class of bundles, we have the following result.

Theorem 4.3 (G, Manon)**.** *If $\mathcal{E}$ is a sparse hypersurface bundle, then $\mathbb{P}(\mathcal{E} \otimes V)$ and $\mathcal{FL}(\mathcal{E})$ are Mori dream spaces for any vector space V.*

We then apply this result to the tangent bundle of a product of projective spaces. Let $\mathcal{T}_n$ denote the tangent bundle of $\mathbb{P}^n$. We have that $\mathcal{T}_n$ is a sparse hypersurface bundle, where D is the $n+1 \times n+1$ identity matrix. By the above result, we have that $\mathbb{P}(\mathcal{T}_n \otimes V)$ and $\mathcal{FL}(\mathcal{T}_n)$ are Mori dream spaces for any vector space V, but this result can be extended further by the following lemma as a straightforward consequence of [KMa, Theorem 1.4].

Lemma 4.4 (G, Manon)**.** *Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be vector bundles over toric varieties $Y(\Sigma_1)$ and $Y(\Sigma_2)$, respectively, and suppose that $\mathbb{P}(\mathcal{E}_1)$ and $\mathbb{P}(\mathcal{E}_2)$ are Mori dream spaces. Then $\mathbb{P}(\mathcal{E}_1 \times \mathcal{E}_2)$ is a Mori dream space, where $\mathcal{E}_1 \times \mathcal{E}_2$ is the product bundle over $Y(\Sigma_1) \times Y(\Sigma_2)$.*

Then, as an extension, we can consider the product of tangent bundles of the product of projective spaces. Let $\mathbf{n} = (n_1, ..., n_m)$ with $n_i > 0$ and let $\mathcal{T}_{\mathbf{n}}$ denote the tangent bundle of $\prod_{i=1}^{m} \mathbb{P}^{n_i}$.

Corollary 4.5 (G, Manon)**.** *For any V, $\mathbb{P}(\mathcal{T}_{\mathbf{n}} \otimes V)$ and $\mathcal{FL}(\mathcal{T}_{\mathbf{n}})$ are Mori dream spaces.*

The remainder of this section details the efforts to give precise presentations of the Cox rings of these bundles, starting with the following result

Proposition 4.6 (G, Manon)**.** *Let $1 \leq m < n$. Then $\mathcal{R}(\mathcal{T}_{\mathbf{n}} \otimes \mathbb{K}^m)$ has the following presentation:*

$$\mathcal{R}(\mathcal{T}_{\mathbf{n}} \otimes \mathbb{K}^m) = \mathbb{K}\left[x_j, Y_{ij} | 1 \leq i \leq m, 0 \leq j \leq n\right] / \left\langle \sum x_j Y_{ij} | 1 \leq i \leq m \right\rangle.$$

In the case that $m = n$, the behavior becomes more interesting, as there is now a need for an additional generator. We use [KMa, Theorem 1.4] and a close relationship with a particular *Zelevinsky quiver variety* (see [MS05, Chapter 17]) to compute the presentation of $\mathbb{R}(\mathcal{T}_{\mathbf{n}} \otimes \mathbb{K}^n)$.

Proposition 4.7. $\mathbb{R}(\mathcal{T}_{\mathbf{n}} \otimes \mathbb{K}^n) = \mathbb{K}[x_j, Y_{ij}, W]/\langle \sum x_j Y_{ij}, \det Y(j) - x_j W \rangle$.

In [KMa, Algorithm 5.6], Kaveh and Manon give a description of the *subduction algorithm*, which builds a finite generating set of the Cox ring, if one exists. (The algorithm will build an infinite generating set, if not a Mori dream space.) In 2019, we implemented this algorithm as code in *Macaulay2* for toric vector bundles, allowing us to not only test if the bundles were Mori dream, but also produce a presentation of their Cox rings. Some examples of the results obtained by these experimental methods can be found in Proposition 4.6 and 4.7.

While the Macaulay2 code we currently have is only suited to compute the Cox rings of toric vector bundles, we believe it can be modified to other spaces. Therefore, as part of the continued work, we would like to adapt the code and run it on other spaces, potentially answering the question of whether $\overline{M}_{0,n}$ is a Mori dream space for $n \in \{7, 8, 9\}$

5 Undergraduate Research

In January 2022, I began working with Dr. Christopher Manon to mentor four (now five) undergraduate students on a research project with a goal of finding a closed form for the Ehrhart series of a class of polytopes formed from trivalent trees. This section details that work.

Let $\mathcal{L}$ ($\cong \mathbb{Z}^m$) $\subset \mathbb{R}^n$ be a lattice. For a polytope $\Delta \subset \mathbb{R}^n$ with vertices in $\mathcal{L}$, the *lattice point enumerator* is the function:

$$E_\Delta(n) = |n\Delta \cap \mathcal{L}|$$

which counts the number of interior lattice points of the n-th dialate of Δ. The lattice point enumerator is known to be a polynomial of degree equal to the dimension d of Δ, so it is also known as the Ehrhart polynomial and may be denoted $\mathrm{Ehr}_\Delta(n)$. The *Ehrhart series* is the formal power series:

$$\mathrm{Ehr}_\Delta(z) := 1 + \sum_{n=1}^{\infty} E_\Delta(n) z^n,$$

It can then be shown that the series $\mathrm{Ehr}_\Delta(z)$ is rational:

$$\mathrm{Ehr}_\Delta(z) = \frac{h_\Delta(z)}{(1-z)^{d+1}}$$

where $h_\Delta(z)$ is a polynomial of degree $\leq d$.

Our focus has been on polytopes derived from *trivalent trees* with L vertices. From these trees, we are able to use inequalities to form a polytope, denoted Δ_L. The work from Summer 2022 culminated in closed-form equations for $\mathrm{Ehr}_{\Delta_L}(z)$, depending on the parity of L.

Current work on the project includes considering a modification of the trivalent tree where the ends are identified, creating a sunbeam shape, which creates a new class of polytopes to consider.

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