RESEARCH STATEMENT

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Introduction

My research focuses on the birational geometry of moduli spaces. A natural question one can ask about a moduli space – or indeed, about any variety – is to describe the rational maps from that space to other projective varieties. This line of inquiry has quickly become a popular area of research, with numerous open problems including the F-Conjecture [KM], the Mori Dream Space Conjecture [HK00], and the Hassett-Keel program [HH09a, HH09b]. In recent years, a new guiding principle has emerged. Often, the image of a moduli space under such a map has a modular interpretation itself. By varying the moduli functor, we can discover a great deal about the birational geometry of a given moduli space.

The main tool I use in my work is geometric invariant theory (GIT). In order to construct a moduli space, one typically begins with a space that parameterizes *embedded* varieties, such as a Hilbert scheme or Chow variety, and then quotients by the automorphism group G of the ambient space. It has long been understood that the GIT quotient is not in fact a quotient of the whole parameter space, but only of an open set, called the *semistable* locus. Moreover, such quotients are not unique – they depend on a choice of a G-linearized ample line bundle. While initially seen as a drawback of the theory, from the modern perspective these issues are actually an advantage. By varying the choice of linearization, we produce a different set of semistable points, and thus obtain many different related moduli spaces.

In my previous research, I have used this GIT approach to study the birational geometry of several spaces. These results are outlined here.

Summary of Previous Research

(1) **The Hassett-Keel Program in Genus Four** [CMJL11, CMJL12]**.** The goal of the Hassett-Keel program is to describe each of the log canonical models of $\overline{M}_g$ as a moduli space parameterizing certain types of curves. As of this writing, only a few stages of the Hassett-Keel program have been carried out for general genus g, but the program has been completed in genus 2 and 3 [Has05, HL10b]. Recently, in joint work with Sebastian Casalaina-Martin and Radu Laza, we nearly complete the program in genus 4, describing the final 7 stages using variation of GIT on a single space.

(2) **Late Stages of the Hassett-Keel Program** [FJ12]**.** In joint work with Maksym Fedorchuk, we study the log canonical model of $\overline{M}_g$ obtained via GIT for second Hilbert points of canonical curves. We provide an explicit criterion for a smooth curve to be semistable, and also identify a vast array of semistable singular curves.

(3) **Effective Divisors on $\overline{M}_{g,1}$** [Jen11a, Jen11b]**.** Using variation of GIT, I have identified extremal rays of the effective cone $\overline{Eff}(\overline{M}_{g,1})$ for all $g \leq 6$. The effective divisors that generate these rays are pointed analogues of the famous Brill-Noether divisors, which are known to generate extremal rays of $\overline{Eff}(\overline{M}_g)$ for similarly small values of g.

(4) **Compactifications of $M_{0,n}$** [GJM12, GJMS12]**.** In joint work with Noah Giansiracusa and Han-Bom Moon, we construct a large number of compactifications of $M_{0,n}$ using variation of GIT on a single space. Each of these compactifications receives a morphism from $\overline{M}_{0,n}$, and in a recent preprint with Angela Gibney, Han-Bom Moon and Dave Swinarski, we identify the corresponding nef divisor classes on $\overline{M}_{0,n}$. In many cases, these divisors are related to vector bundles of conformal blocks on $\overline{M}_{0,n}$.

(5) **A Tropical Approach to the Gieseker-Petri Theorem** [BJM+12]**.** Two of the most celebrated theorems in modern algebraic geometry concern the variety $\mathcal{G}^r_d(X)$, which parameterizes linear series of a given degree and rank on a smooth projective curve X. The first of these is the Brill-Noether

Theorem, which says that $\mathcal{G}^r_d(X)$ has the expected dimension for sufficiently general curves X [GH80] [EH83] [Laz86], and the second is the Gieseker-Petri Theorem, which says that $\mathcal{G}^r_d(X)$ is smooth for sufficiently general X [Gie82] [EH83]. Recently, a team consisting of Cools, Draisma, Payne and Robeva provided an independent proof of the Brill-Noether Theorem using techniques from tropical geometry [CDPR]. This past semester, I led a small group of undergraduates towards a proof of the r = 1 case of the Gieseker-Petri Theorem [BJM+12].

Background

This section is intended for those readers who are less familiar with algebraic geometry. More details on my previous and proposed research can be found in Sections 1 through 4.

Algebraic geometers are interested in varieties – that is, solution sets to systems of polynomial equations. One of the fundamental motivating problems in algebraic geometry is to classify all varieties. To begin, we may attempt to classify varieties of a given dimension. Zero-dimensional varieties are just points, so the first interesting case is that of curves. Over the complex numbers, these are the same as Riemann surfaces, or topological surfaces equipped with a complex structure. The underlying topological space is determined by its genus, and we may therefore further restrict our problem: how can we describe the set of all curves of a given genus g?

We define $M_{g,n}$ to be the set of all isomorphism classes $(C, p_1, \ldots, p_n)$ where C is a smooth curve of genus g and the p_i's are distinct points of C. (Note: when $n = 0$, it is standard to write M_g rather than $M_{g,0}$.) This set can be endowed with a natural geometric structure, so that smoothly varying families of curves correspond to smoothly varying points in $M_{g,n}$. The space also has a natural compactification, denoted $\overline{M}_{g,n}$, which parameterizes certain types of mildly singular curves.

Moduli spaces such as this one provide us with a sort of meta-geometric perspective – studying their geometry tells us about properties of varieties and how they vary in families. For this reason, moduli spaces have far-reaching applications to fields as diverse as number theory and mathematical physics. A possible approach to studying $\overline{M}_{g,n}$ would be to describe the set of morphisms it admits to other projective varieties, and a natural generalization is to consider its *rational maps*, or morphisms defined on a dense open subset. If the rational map $f : \overline{M}_{g,n} \dashrightarrow Y$ has an inverse, it is called a *birational map*, and we refer to Y as a *birational model* for $\overline{M}_{g,n}$. The majority of my research focuses on constructing birational models for $\overline{M}_{g,n}$ and other moduli spaces.

1. Hassett-Keel Program

The Hassett-Keel program aims to give modular interpretations of certain log canonical models of $\overline{M}_g$, with the ultimate goal of giving a modular interpretation of the canonical model in the case $g \gg 0$. This program, while relatively new, has attracted the attention of a number of researchers, and has rapidly become one of the most active areas of research concerning the moduli of curves. More specifically, for $\alpha \in [0, 1]$, the log minimal models of $\overline{M}_g$ are defined to be the projective varieties

$$\overline{M}_g(\alpha) := \operatorname{Proj}\left(\bigoplus_{n=0}^{\infty} H^0\left(n(K_{\overline{M}_g} + \alpha\Delta)\right)\right),$$

where Δ is the boundary divisor in $\overline{M}_g$. For $\alpha \geq \frac{7}{10} - \epsilon$, Hassett and Hyeon have shown that the log minimal models $\overline{M}_g(\alpha)$ are themselves moduli spaces parameterizing certain types of curves [HH09a, HH09b]. The idea of the Hassett-Keel program is that this phenomenon should extend to all values of α. As we decrease α from 1 to 0, we should be able to progressively describe each new model $\overline{M}_g(\alpha)$ as a moduli space until we reach the canonical model $\overline{M}_g(0)$.

1.1. **The Hassett-Keel Program in Genus Four.**

In a recent series of papers, Hassett and Hyeon provide modular descriptions of the first few stages of the log minimal model program for $\overline{M}_g$ [HH09a, HH09b]. In particular, they identify $\overline{M}_g(\frac{9}{11})$ with the moduli space $\overline{M}_g^{ps}$ of pseudostable curves constructed by Schubert [Sch91], in which curves with elliptic tails – that is, a subcurve of genus 1 meeting the rest of the curve in one point – are replaced with cuspidal curves. The next stage of the log minimal model program, also described by Hassett and Hyeon [HH09b], occrs at the critical slope $\alpha = \frac{7}{10}$, where we observe a flip of $\overline{M}_g^{ps}$. Roughly speaking, this rational map replaces elliptic bridges with tacnodes. As of this writing, this first divisorial contraction and first flip are the only known stages of the Hassett-Keel program that have been described for general genus g. For specific, small values of g, more is known. The cases of $g = 2$ and 3, in which the full programs consist of only one stage more than those described above, were completed by Hassett and Hyeon-Lee, respectively [Has05, HL10b]. In the genus 4 case, one more model was constructed by Hyeon and Lee [HL10a], and in my own joint work with Sebastian Casalaina-Martin and Radu Laza, we use variation of GIT to nearly complete the program in this case, describing 7 additional stages [CMJL11, CMJL12].

Theorem 1.1. [CMJL12] *For $\alpha \leq \frac{5}{9}$, the log minimal models $\overline{M}_4(\alpha)$ arise as GIT quotients of a single parameter space.*

One of the models we construct is the first example of a flip of the hyperelliptic locus in $\overline{M}_g$, which is expected to exist for all genera. In this case we see that the rational map from $\overline{M}_4$ to this space contracts the hyperelliptic locus to a point, corresponding to twice the cycle of a rational normal curve. The fiber over this point via the next map in the program consists of curves with A_{2g} singularities, as predicted in my joint work with Maksym Fedorchuk [FJ12]. As a result of this project, we have developed a great deal of insight about the problem in the general case.

1.2. **Finite Hilbert Stability.** A recent development in the Hassett-Keel program involves the GIT analysis of finite (i.e. non-asymptotic) Hilbert stability of canonical curves. Let $C \hookrightarrow \mathbb{P}^{g-1}$ be a canonical curve, and for each integer $m \geq 2$, define the m^{th} Hilbert point $[C]_m$ of C to be the space of m-forms vanishing on C. In other words,

$$[C]_m := H^0(I_C(m)) \subset Gr\left(\binom{g+m-1}{m} - (2m-1)(g-1), H^0(\mathbb{P}^{g-1}, \mathcal{O}(m))\right).$$

We denote by $\overline{Hilb}^m_{g,1}$ the closure of the locus of m^{th} Hilbert points of canonical curves in the Grassmannian above. The variety $\overline{Hilb}^m_{g,1}$ comes equipped with a natural linearization $\mathcal{O}(1)$ coming from the Plucker embedding of the Grassmannian. It is widely expected that the GIT quotients $\overline{Hilb}^m_{g,1}/\!\!/SL(g)$ will yield further stages of the Hassett-Keel program [Mor09] [MS11]. An important recent development in this area is the work of Alper, Fedorchuk and Smyth [AFS12], where it is shown that finite Hilbert points of general canonical curves are semistable in all genera. In joint work with Sebastian Casalaina-Martin and Radu Laza, we describe the GIT quotients $\overline{Hilb}^m_{4,1}/\!\!/SL(4)$ for all m and show that they yield the final seven stages of the Hasset-Keel program in genus four [CMJL12]. In joint work with Maksym Fedorchuk [FJ12], we study the space $\overline{Hilb}^2_{g,1}$ in detail, providing an explicit criterion for a point to be semistable. To date, this is the only known example of such a criterion for finite Hilbert stability in general genus.

Theorem 1.2. [FJ12] *A canonical curve of genus g not lying on a quadric of rank 3 or less has semistable 2^{nd} Hilbert point.*

1.3. **Late Stages of the Hassett-Keel Program via Syzygies.** A significant drawback to the finite Hilbert approach is that it is only capable of producing the log minimal models $\overline{M}_g(\alpha)$ for $\alpha \geq \frac{g+12}{7g+6}$. While the study of these quotients could significantly improve our understanding of the Hassett-Keel program, it cannot possibly complete the program for sufficiently large g. Other suggested approaches to the Hassett-Keel program suffer a similar fate. The stack-theoretic approach pioneered by Alper, Smyth and Van der Wyck [ASVdW10], for example, promises to construct the log minimal models $\overline{M}_g(\alpha)$ for $\alpha \geq \frac{3}{8}$. We hope to generalize the finite Hilbert construction using syzygies of canonical curves.

The idea, which was developed in conversations with Sean Keel, is to define $Z^d_2(C) = H^0(I_C(d+1))$ to be the $(d+1)^{th}$-Hilbert point of C, and inductively define $Z^d_k(C)$ as the kernel of the multiplication map

$$Z^1_{k-1}(C) \otimes H^0(\mathbb{P}^{g-1}, \mathcal{O}(d)) \to Z^{d+1}_{k-1}(C).$$

So, for example, $Z^1_3(C)$ is the space of linear syzygies between the quadrics containing C. We will call $Z^1_k(C)$ the k^{th} syzygy point of C, and we further define $\overline{Syz}^k_{g,1}$ to be the closure of the locus of k^{th} syzygy points of canonical curves in a similar Grassmannian.

We suspect that late stages of the Hassett-Keel program will coincide with the GIT quotients $\overline{Syz}^k_{g,1} /\!\!/ SL(g)$. More concretely, we expect that:

$$\overline{Syz}^k_{g,1} /\!\!/ SL(g) \cong \overline{M}_g \left(\frac{g^2 - 3kg + 17g - 8k + 4}{7g^2 - 8kg + 15g - 4k + 2} \right).$$

For $g \gg 0$ and $k = \lfloor \frac{g}{2} \rfloor$, this α-value is negative, hence we are obtaining models that lie *past* the canonical model in the Hassett-Keel program. Indeed, when g is even, the slope of $K_{\overline{M}_g} + \alpha\Delta$ with α as above is precisely that of the Gieseker-Petri divisor, which was famously shown in [EH87] to be smaller than that of the canonical class (see §2.1 for more details). It is therefore reasonable to hope that, together with the finite Hilbert construction, the syzygies of canonical curves could yield every stage of the Hassett-Keel program for general g.

We note that the spaces $\overline{Syz}^2_{g,1}$ and $\overline{Hilb}^2_{g,1}$ agree – that is, the latest possible stage that can be constructed via finite Hilbert points is the earliest possible stage than can be constructed via syzygies. We hope to extend Theorem 1.2 to the spaces $\overline{Syz}^k_{g,1}$ for all k, showing that the general canonical curve is semistable in each case. Such a result would be a natural analogue of that on finite Hilbert stability due to Alper, Fedorchuk and Symth [AFS12]. In particular, we expect that:

Conjecture 1.3. *A canonical curve of genus g not lying on a degenerate rational normal scroll of degree k has semistable k^{th} syzygy point.*

Interestingly, this condition has an intrinsic interpretation in terms of linear series on a curve. Specifically, the canonical embedding of a curve C will be contained in such a scroll if and only if the variety $\mathcal{G}^1_d(C)$ parameterizing linear series of rank 1 and degree d on C is singular for some $d \leq g + 1 - k$.

We note one other intriguing consequence of this syzygy construction. Green's Conjecture, which is known to hold for generic curves [Voi02] [Voi05], suggests that the set of curves of low Clifford index is contained in the base locus of the rational map $\varphi_k : \overline{M}_g \dashrightarrow \overline{Syz}^k_{g,1} /\!\!/ SL(g)$. Indeed, in [FJ12], we show that hyperelliptic curves, bielliptic curves, and trigonal curves with positive Maroni invariant are contained in the base locus of φ_2. On the other hand, the base locus of the rational map $\overline{M}_g \dashrightarrow \overline{M}_g(\alpha)$ is known to be a union of rationally connected varieties for all $\alpha \geq 0$. Combining these ideas, we conjecture the following.

Conjecture 1.4. *The locus of d-gonal curves in $\overline{M}_g$ is rationally connected for all $d < \frac{1}{3}g$.*

We note that, to date, the best known bound for d as above is independent of the genus g.

2. Cones of Divisors on Moduli Spaces of Curves

As described in the introduction, a central motivating question in this field is to describe birational models of $\overline{M}_g$. If Y is such a birational model, and the birational map $f : \overline{M}_g \dashrightarrow Y$ is a contraction, then for any ample line bundle L_Y on Y we have

$$Y = \mathrm{Proj} R(\overline{M}_g, f^* L_Y),$$

where $R(X, L)$ is the section ring

$$R(X, L) := \bigoplus_n H^0(X, L^{\otimes n}).$$

In this way, we see that the birational model Y is canonically associated to the effective bundle f^*L_Y, and we are naturally led to studying effective line bundles on $\overline{M}_g$. The set of effective divisors forms a cone in the Neron-Severi spaces whose closure $\overline{Eff}(\overline{M}_g)$ is called the *pseudoeffective cone.* Similarly, if the rational map $f : \overline{M}_g \dashrightarrow Y$ is regular, then the bundle f^*L_Y is nef. In this way, the problem of describing morphisms from $\overline{M}_g$ to other projective varieties is intimately related to that of describing the nef cone $Nef(\overline{M}_g)$.

2.1. **Effective Divisors on Moduli Spaces of Pointed Curves.** The past 30 years have witnessed a great deal of research on $\overline{Eff}(\overline{M}_g)$, particularly on effective divisors contained in the span of the boundary divisor Δ and the canonical divisor $K_{\overline{M}_g}$. Among the many good reasons for this is the well-known series of papers in which Harris-Mumford and Eisenbud-Harris prove that the moduli space $\overline{M}_g$ is of general type for $g \geq 24$ [HM82] [EH87]. A key element of their proof is the fact that, for these same values of g, there exists an $\alpha > 0$ such that $K_{\overline{M}_g} - \alpha\Delta$ is effective. This work led to the Harris-Morrison Slope Conjecture, which also concerns effective divisors in the $(K_{\overline{M}_g}, \Delta)$-plane, and was recently demonstrated to be false in [FP05] and [Far09]. Another line of work that focuses on divisors in this plane is the Hassett-Keel program, described in §1, whose goal is to identify the Mori chambers of $\overline{Eff}(\overline{M}_g)$ that intersect this plane.

Today, extremal rays of the effective cone $\overline{Eff}(\overline{M}_g)$ have been identified for all $g \leq 11$. In particular, the locus $BN^r_d \subset \overline{M}_g$, consisting of curves admitting a linear series of rank r and degree d, is a divisor whenever $g - (r+1)(g-d+r) = -1$. This divisor, called a *Brill-Noether divisor*, is known to generate an extremal ray of $\overline{Eff}(\overline{M}_g)$ whenever $g \leq 11$ and $g+1$ is composite. In the remaining cases ($g = 4, 6$ and 10), there are other divisors that are known to generate extremal rays.

In [Jen11a, Jen11b], I initiate a program for studying effective divisors on the moduli space $\overline{M}_{g,1}$ for small values of g using variation of GIT. In particular, I show that this cone has extremal rays generated by certain pointed analogues of Brill-Noether divisors that were previously studied in [Log03]. These divisors parameterize pointed curves admitting a linear series with unusual ramification at the marked point.

Theorem 2.1. [Jen11a, Jen11b] *For all $g \leq 5$, there is an extremal ray of $\overline{Eff}(\overline{M}_{g,1})$ generated by a pointed Brill-Noether divisor. Moreover, there is an extremal ray of $\overline{Eff}(\overline{M}_{6,1})$ generated by the divisor $\mathcal{D}_6$ of "nodes of g^2_6's".*

This work has potential applications to the effective cone of $\overline{M}_g$. While the intersection of $\overline{Eff}(\overline{M}_g)$ with the $(K_{\overline{M}_g}, \Delta)$-plane has been described for all $g \leq 11$, a complete description of $\overline{Eff}(\overline{M}_g)$ has only been worked out in the cases of $g = 2$ and 3 [Rul01]. A natural conjecture is the following.

Conjecture 2.2. *For every genus $g \leq 11$, the pseudoeffective cone $\overline{Eff}(\overline{M}_g)$ is simplicial, generated by the boundary divisors Δ_i and:*

(1) *The Brill-Noether divisor BN^r_d when $g \neq 4, 6$ or 10;*
(2) *The Gieseker-Petri divisor GP^1_d when $g = 4$ or 6;*
(3) *The K3 divisor when $g = 10$.*

We note that each of these divisors is known to generate an extremal ray of $\overline{Eff}(\overline{M}_g)$. The content of this conjecture is that there are no other extremal rays. To approach this problem, we propose to adapt an argument from [Rul01] to these higher-genus cases. Specifically, we will try to determine faces of the pseudoeffective cone by pulling back divisors via the gluing maps:

$$\overline{M}_{g-1,2} \to \Delta_0 \subset \overline{M}_g$$

$$\overline{M}_{i,1} \times \overline{M}_{g-i,1} \to \Delta_i \subset \overline{M}_g.$$

In this way, we reduce the problem to that of studying $\overline{Eff}(\overline{M}_{k,n})$ for $k < g$ and $n = 1, 2$. The divisors that I study in [Jen11a, Jen11b] are pullbacks of Brill-Noether divisors via these maps, which suggests that my previous research will apply directly.

The first case where this question is open is that of genus 4, where it is related to the famous question of whether there exist projective surfaces in M_4 [Dia84]. To see this, we briefly describe a proof that there are no projective surfaces in M_3. Note that, if M_3 contains a projective surface, then its intersection with the hyperelliptic divisor $BN^1_2 \subset \overline{M}_3$ is a projective curve that does not meet the boundary in BN^1_2. But the pseudoeffective cone $\overline{Eff}(BN^1_2)$ is generated by boundary classes, so this is impossible. In the case of genus 4, the Gieseker-Petri divisor $GP^1_3 \subset \overline{M}_4$ plays a role similar to that of the hyperelliptic divisor in genus 3. In this way, we see that a description of $\overline{Eff}(GP^1_3)$ would not only yield a generalization of Rulla's argument to genus 4, but could potentially resolve this well-known conjecture as well.

2.2. **Nef Divisors on $\overline{M}_{0,n}$.** In [GKM02], Gibney, Keel and Morrison show that the nef cone of $\overline{M}_g$ is completely determined by that of $\overline{M}_{0,g}$. For this reason, the bulk of research on nef divisors on moduli spaces of curves has focused on the genus zero case. The moduli space $\overline{M}_{0,n}$ of Deligne-Mumford stable n-pointed rational curves is a natural compactification of $M_{0,n}$. The problem of describing nef divisors on $\overline{M}_{0,n}$ is intimately related to that of describing alternate compactifications of $M_{0,n}$, which has rapidly become a central project in this field [Kap93a, Kap93b, Bog99, LM00, Has03, Smy09, Fed11, GS10, Gia11, GJM12]. In recent joint work with Noah Giansiracusa and Han-Bom Moon, we develop a GIT approach to such constructions that generalizes and unifies many of the prior approaches that appear in the literature [GJM12].

We explain our construction here. The Chow variety of degree d curves in $\mathbb{P}^d$ has an irreducible component parameterizing rational normal curves and their limit cycles. Denote this by $Chow(1, d, \mathbb{P}^d)$ and consider the locus

$$U_{d,n} := \{(X, p_1, \ldots, p_n) \in Chow(1, d, \mathbb{P}^d) \times (\mathbb{P}^d)^n \mid p_i \in X \ \forall i\}.$$

There is a natural action of $\mathrm{SL}(d+1)$ on $U_{d,n}$, and the main objects we study are the GIT quotients $U_{d,n} /\!\!/ \mathrm{SL}(d+1)$. These depend on a linearization $L \in \mathbb{Q}^{n+1}_{>0}$ which can be thought of as assigning a rational weight to the curve and another weight to each of its marked points. Each linearization determines a different quotient, and the set of different possible quotients partitions the GIT cone into chambers.

A preliminary stability analysis reveals that a rational normal curve with distinct marked points is stable for an appropriate range of linearizations, so each of the GIT quotients in question yields a compactification of $M_{0,n}$. We go on to determine the remaining GIT-stable points of $U_{d,n}$ for each such linearization, thus describing a large set of alternate compactifications that we call *Veronese quotients*. These quotients are remarkable in that they specialize to nearly every known compactification of $M_{0,n}$, including Hassett's weighted spaces [Has03] and the Kontsevich-Boggi compactification [Bog99]. Moreover, each Veronese quotient is projective, and each is the image of a regular map defined on $\overline{M}_{0,n}$, so each provides us with a collection of nef divisors on $\overline{M}_{0,n}$.

The $d = 1$ case of our construction yields the GIT quotients $(\mathbb{P}^1)^n /\!\!/ \mathrm{SL}(2)$, which have made numerous appearances in the literature, including Kapranov's result that their inverse limit is $\overline{M}_{0,n}$ [Kap93a]. They are even included in Mumford's book [MFK94] as "an elementary example" of GIT. The paper [GS10] introduces and investigates Veronese quotients in the $d = 2$ case. In [Gia11], Giansiracusa introduces and studies GIT quotients parameterizing the configurations of points in projective space that arise in $U_{d,n}$, for $1 \leq d \leq n-3$. These can be viewed as a special case of a Veronese quotient obtained by setting the linearization on the Chow factor to be trivial. In fact, Veronese quotients appear to include as special cases all GIT quotients of pointed rational curves that have previously been studied.

Veronese quotients promise to provide us with significant insight into the nef cone $Nef(\overline{M}_{0,n})$. Every projective GIT quotient comes equipped with a natural choice of ample divisor, and the pullback of such a divisor to $\overline{M}_{0,n}$ is nef. In a recent paper, joint with Angela Gibney, Han-Bom Moon, and Dave Swinarksi [GJMS12], we compute the class of these nef divisors on $\overline{M}_{0,n}$. Moreover, the chamber decomposition of the GIT cone mirrors the structure of the nef cone, suggesting that the GIT walls of highest codimension ought to correspond to extremal rays of $Nef(\overline{M}_{0,n})$. Our hope is to use this construction, together with our class computation, to identify such extremal rays.

A recent development in the birational geometry of $\overline{M}_{0,n}$ involves divisors that arise from conformal field theory. These divisors are first Chern classes of vector bundles of conformal blocks on the moduli stack $\overline{\mathcal{M}}_{g,n}$. Constructed using the representation theory of affine Lie algebras [Tsu, Fak12], these vector bundles depend on the choice of a simple Lie algebra $\mathfrak{g}$, a positive integer ℓ, and an n-tuple $\vec{\lambda} = (\lambda_1, \cdots, \lambda_n)$ of dominant integral weights in the Weyl chamber for $\mathfrak{g}$ of level ℓ. Vector bundles of conformal blocks are globally generated when $g = 0$ [Fak12], and their first Chern classes $c_1(\mathbb{V}(\mathfrak{g}, \ell, \vec{\lambda})) = \mathbb{D}(\mathfrak{g}, \ell, \vec{\lambda})$, the conformal block divisors, are nef. In [GJMS12], we describe a family of Veronese quotients that correspond to divisors of conformal blocks on $\overline{M}_{0,n}$. This is reminiscent of the work of Giansiracusa [Gia11], who shows that every linearization with trivial Chow factor gives rise to a conformal blocks divisor on $\overline{M}_{0,n}$. Together, these results suggest a correlation between Veronese quotients, conformal blocks, and extremal rays of $Nef(\overline{M}_{0,n})$.

3. Tropical Approaches to Studying Linear Series on Curves

A central object in the study of algebraic curves is the variety of linear series on a curve. Given a smooth projective curve X, we write $\mathcal{G}^r_d(X)$ for the variety parameterizing linear series of degree d and rank r on X. The nature of this variety for general curves is the central focus of two of the most celebrated theorems in modern algebraic geometry.

Brill-Noether Theorem. [GH80] *If X is a general curve of genus g, then dim $\mathcal{G}^r_d(X) = \rho = g - (r+1)(g - d + r)$. If $\rho < 0$, then $\mathcal{G}^r_d(X)$ is empty.*

Gieseker-Petri Theorem. [Gie82] *If X is a general curve, then $\mathcal{G}^r_d(X)$ is smooth.*

These theorems differ from more classical results such as Riemann-Roch in that they concern *general*, rather than arbitrary, curves. As such, the original proofs due to Griffiths-Harris [GH80] and Gieseker [Gie82] make use of degeneration techniques. These ideas were later refined by Eisenbud and Harris [EH83], giving a second proof of both theorems. A subsequent proof, due to Lazarsfeld, avoids using degeneration arguments by working instead with curves on a K3 surface [Laz86].

More recently, a team consisting of Cools, Draisma, Payne and Robeva provided an independent proof of the Brill-Noether Theorem using techniques from tropical geometry [CDPR]. More specifically, they use the theory of divisors on metric graphs, as developed by Baker and Norine in [BN07], to construct a Brill-Noether general graph Γ_g with first Betti number g. Combining this with Baker's Specialization Lemma [Bak08], which says that the rank of a divisor on a smooth curve over a discretely valued field jumps under specialization to the dual graph of the central fiber, they obtain a new proof of the Brill-Noether Theorem.

This past semester, I led a team of undergraduates to prove the $r = 1$ case of the Gieseker-Petri Theorem using a similar approach [BJM+12]. In other words, we show that $\mathcal{G}^1_d(X)$ is smooth for the general curve X of arbitrary genus. To do this, we use the same metric graph Γ_g that appears in [CDPR]. This graph, depicted below, consists of g loops arranged in a chain. The smoothness of $\mathcal{G}^r_d(X)$ at a linear series $W \subset H^0(X, L)$ is equivalent to the condition that the multiplication map $W \otimes H^0(X, K_X - L) \to H^0(X, K_X)$ is injective. In the rank 1 case, the kernel of this map is naturally isomorphic to $H^0(X, K_X - 2L)$, so it suffices to prove:

Theorem 3.1. [BJM+12] *The graph Γ_g does not admit a positive-rank divisor D such that $K_{\Gamma_g} - 2D$ is linearly equivalent to an effective divisor.*

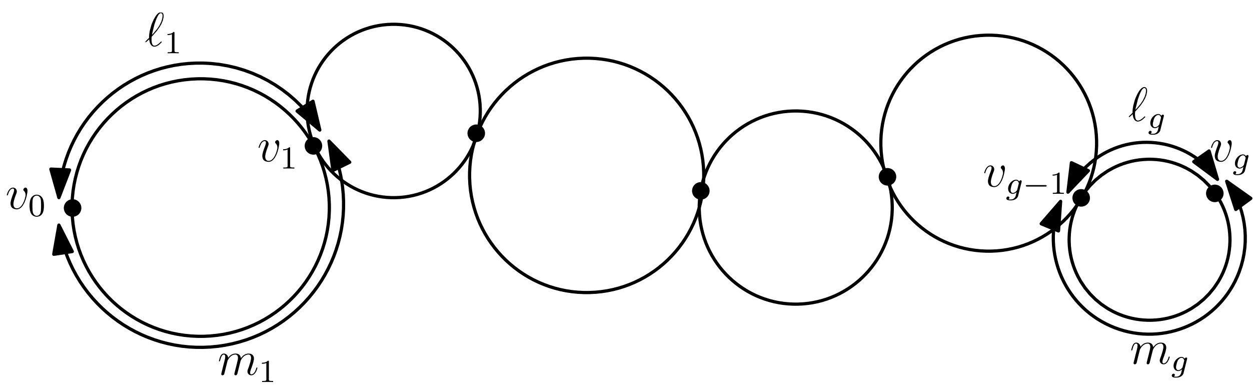

FIGURE 1. The graph Γ_g from [CDPR]

Recent conversations with Sam Payne have yielded an approach toward the general case of the Gieseker-Petri Theorem. As in the earlier arguments, we suppose that there is a regular family of curves X over a DVR whose special fiber has dual graph Γ_g. The idea springs from two basic observations. First, given a line bundle L on X, one can choose a basis for $H^0(X, L)$ such that the specialization of each basis element yields a natural choice of chip configuration on Γ_g. Second, if $\sum \alpha_{ij}(s_i \otimes t_j) \in H^0(X, L) \otimes H^0(X, M)$ lies in the kernel of the multiplication map, then at each point p of the central fiber, the minimal order of vanishing among the $s_i \otimes t_j$ appearing with nonzero coefficient must occur at least twice. These orders of vanishing are precisely what is encoded by the chip configurations.

In the case that $M = K_X - L$, this approach could yield a new proof of the Gieseker-Petri Theorem, but one could easily study more general multiplication maps using the same technique. Such a theory, if developed, could potentially be used to not only provide tropical proofs of known theorems, such as Green's Conjecture for the general curve (see [Voi02] and [Voi05]), but also to shed light on open questions like the Maximal Rank Conjecture.

4. Moduli Spaces of Higher-Dimensional Varieties

In contrast to the theory of curves, very little is known about moduli spaces of higher-dimensional varieties. While the majority of my work thus far has focused on moduli of curves, Geometric Invariant Theory techniques are equally applicable to higher-dimensional moduli problems. The natural analogue of $M_{g,n}$ to higher dimensions is the moduli space of pairs (X, H) where X is a smooth variety and H is a divisor on X with simple normal crossings. A natural compactification of this space, akin to the compactification $M_{g,n} \subset \overline{M}_{g,n}$, was described by Kollár-Shepherd-Barron [KSB88] and Alexeev [Ale96]. This compactification parameterizes pairs (X, H) such that $(X, \epsilon H)$ has semi-log canonical singularities. In this section we discuss two current projects that use GIT to study higher-dimensional varieties. More specifically, we discuss higher-dimensional analogues of the Hassett-Keel program, and how these shed light on the more general moduli problem.

4.1. **Moduli of Kummer Varieties.** Our initial motivation for studying Kummer varieties comes from the theory of curves. As described in the introduction, a natural question about $\overline{M}_g$ is to ask what maps it admits to other varieties, and such maps are of particular interest if they have a modular interpretation. The idea of this project is to use the geometry of Kummer varieties to construct a new compactification $\overline{M}_g^{Kum}$ of M_g together with a map

$$Kum : \overline{M}_g \to \overline{M}_g^{Kum},$$

and to examine the relationship between this map and other known compactifications of M_g.

One approach to constructing birational models for $\overline{M}_g$ is the Hassett-Keel program, described in §1, while another approach involves extensions of the Torelli map. Our goal is to construct a moduli space $\overline{M}_g^{Kum}$ that fits neatly between these two stories, but first we must describe the extended Torelli maps in greater detail. The Torelli map is the morphism $t : M_g \to A_g$ that sends a curve X to the pair $(Jac_{g-1}(X), \Theta)$, where $Jac_{g-1}(X)$ is the Jacobian parameterizing line bundles of degree $g-1$ on X, and Θ is the divisor of effective bundles:

$$\Theta = \{\mathcal{L} \in Jac_{g-1}(X) | h^0(X, \mathcal{L}) = h^1(X, \mathcal{L}) \neq 0\}.$$

The moduli space A_g admits many different compactifications. It is natural to ask for a compactification $\overline{A}_g$ such that the Torelli map extends to a morphism from $\overline{M}_g$ to $\overline{A}_g$. The earliest result of this type was the discovery of a map from $\overline{M}_g$ to the Satake compactification $\overline{A}_g^{Sat}$ [Nam73]. Geometrically, this map sends a singular curve X to the Jacobian of its normalization $Jac_{g-1}(\widetilde{X})$. We will write $\overline{M}_g^{Sat}$ for the image of $\overline{M}_g$ under this map.

It was Mumford who first observed that $\overline{M}_g$ admits a map to the second Voronoi compactification $\overline{A}_g^{Vor}$. The proof, which can be found in [Nam76], is purely combinatorial, but it was later given a modular description by Alexeev in [Ale04]. Alexeev defines the canonical compactified Jacobian $Jac_{g-1}(X)$ of a Deligne-Mumford stable curve X and shows that the map

$$t^{Vor} : \overline{M}_g \to \overline{A}_g^{Vor}$$

sends such a curve X to the pair $(Jac_{g-1}(X), \Theta)$. We will write $\overline{M}_g^{Vor}$ for the image $t^{Vor}(\overline{M}_g)$.

Recently, Alexeev and Brunyate constructed a map from $\overline{M}_g$ to the first Voronoi compactification $\overline{A}_g^{Vor(1)}$ and showed that the image is in fact isomorphic to $\overline{M}_g^{Vor}$ [AB11]. In a similar vein, Gibney has found a numerical expression for the pullback of an ample class via any extension of the Torelli map. This result implies that, if the Torelli map extends to a morphism from $\overline{M}_g$ to a toroidal compactification of A_g, then the image is, up to Stein factorization, isomorphic to $\overline{M}_g^{Vor}$ [Gib11]. Together, the results of Gibney and Alexeev-Brunyate suggest an intriguing difference between M_g and A_g – while there are infinitely many different toroidal compactifications of A_g, the compactification of the Jacobian locus appears to be unique.

It is further shown in [Gib11] that the divisor $12\lambda - \Delta_0$ on $\overline{M}_g$ is the pullback of a nef divisor from $\overline{M}_g^{Vor}$. This divisor appears in many different places in the literature [GKM02, SB06, HH09a, Gib11]. Most notably, it is also the pullback of a nef divisor from Schubert's moduli space of pseudostable curves $\overline{M}_g^{ps}$, described in §1, and it is unknown whether or not this divisor is semi-ample. If it is semi-ample, then there is necessarily a variety, which we suggestively call $\overline{M}_g^{Kum}$, and a morphism $\overline{M}_g \to \overline{M}_g^{Kum}$, which factors through both the extended Torelli map and the first stage of the Hassett-Keel program, as in the diagram below.

$$\begin{tikzcd} \overline{M}_g \arrow[r] \arrow[d] \arrow[dr] & \overline{M}_g^{Vor} \arrow[d] \\ \overline{M}_g^{ps} \arrow[r] & \overline{M}_g^{Kum} \end{tikzcd}$$

We propose to construct $\overline{M}_g^{Kum}$ in a manner similar to the construction of $\overline{M}_g$ described in §1. Specifically, we will think of a principally polarized abelian variety X, mapped to projective space via the linear series $|n\Theta_X|$, as a point in a Chow variety. We will write $Chow_n$ for the irreducible component of the Chow variety parameterizing the images of such abelian varieties, and consider the GIT quotient $Chow_n /\!\!/ SL(n^g)$. As an immediate consequence of a result of Kempf [Kem78], we see that, for all $n \geq 2$, the image of every smooth abelian variety is GIT stable. In other words, the rational map $\overline{M}_g \dashrightarrow Chow_n /\!\!/ SL(n^g)$ restricts to a regular map on the locus of curves of compact type M_g^{ct}. We would like to know whether this map extends to the full boundary of $\overline{M}_g$.

While this question is interesting for all n, we are primarily interested in the case that $n = 2$. This is because, when X is indecomposable (that is, X is not the product of two smaller-dimensional abelian varieties), its image under the linear series $|2\Theta_X|$ is its Kummer variety $Kum(X)$. Since an abelian variety can be recovered from its Kummer variety, we see that the map $M_g \to Chow_2 /\!\!/ (2^g)$ is an isomorphism onto its image. On the other hand, if $X = X' \times E$ is the product of an abelian variety of dimension $g-1$ and an elliptic curve, then its image under the linear series $|2\Theta_X|$ is twice the cycle $Kum(X') \times \mathbb{P}^1$. In other words, the map $M_g^{ct} \to Chow_2 /\!\!/ SL(2^g)$ forgets the j-invariants of elliptic tails, and thus factors through the moduli space of pseudostable curves $\overline{M}_g^{ps}$.

4.2. **Moduli of K3 Pairs.** At this point, very few of the ideas described above have been developed in the case of higher-dimensional varieties. Consider, for example, the moduli space $\mathcal{P}_{2d}$ of pairs (X, H) where X is a K3 surface and H is an ample divisor of degree 2d. A natural compactification $\overline{\mathcal{P}}_{2d}$ of $\mathcal{P}_{2d}$ was developed by Kollár-Shepherd-Barron [KSB88] and Alexeev [Ale96] parameterzing so-called stable pairs – that is, pairs (X, H) such that $(X, \epsilon H)$ has semi-log canonical singularities. While this compactification is "nice" from a moduli theoretic point of view, many basic questions about it remain wide open. No one knows, for example, how many boundary components this space has, or what its Picard group is.

One should note the dependence on ϵ in the definition of stable pairs. That is, as α varies from 0 to 1, the moduli space $\overline{\mathcal{P}}_{2d}(\alpha)$ of pairs (X, H) such that $(X, \alpha H)$ has semi-log canonical singularities changes. When α is small, the K3 surface X has only mild singularities, but the curve H may be wildly singular. Conversely, when $\alpha = 1$, the K3 surface may have complicated singularities, but the curve is necessarily Deligne-Mumford stable, and thus $\overline{\mathcal{P}}_{2d}(1)$ admits a natural map to the moduli space of curves $\overline{M}_{d+1}$, given by forgetting the surface. A natural question, first posed by Kollár, is to identify the critical values of α at which the changes occur. This is quite similar to the Hassett-Keel program, described in §1, in which one has a collection of moduli spaces parameterized by a value α between 0 and 1, and the central question is to describe these spaces and the values of α at which they change.

We are currently studying these spaces for small values of d, as an approach toward understanding the more general picture. In a recent paper, Laza describes two alternate compactifications of $\mathcal{P}_2$ using ideas from geometric invariant theory [Laz12]. We briefly describe the approach. A generic K3 surface of degree 2 is a double cover of $\mathbb{P}^2$ branched along a sextic. As such, the moduli space $\overline{\mathcal{P}}_2$ is birational to the GIT quotient

$$\mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}(6)) \times \mathbb{P}^{2\vee} /\!\!/ SL(3).$$

As described in the introduction, this GIT quotient is not unique – it depends upon a choice of an ample line bundle. If the linearization is chosen so that there is a lot of weight on the surface, one obtains the moduli space $\hat{\mathcal{P}}_2$ described in [Laz86]. This moduli space is related to the KSBA compactification $\overline{\mathcal{P}}_2$ by first blowing up the point corresponding to a triple conic, and then composing this with a flip.

By varying the choice of linearization, we obtain a large number of birational models for $\overline{\mathcal{P}}_2$. In the end, we find that, as the linearization varies, these models interpolate between the space $\hat{\mathcal{P}}_2$ described in [Laz12] and a model in which the surface X may be highly singular, but the ample divisor H is forced to be a Deligne-Mumford stable curve. Moreover, the chamber decomposition of the GIT cone should correspond precisely with the Hassett-Keel analogue suggested by Kollár – that of finding the values of α at which the moduli space $\overline{\mathcal{P}}_2(\alpha)$ changes. As a consequence, we hope to describe the pseudoeffective cone $\overline{Eff}(\overline{\mathcal{P}}_2)$ and begin a study of its Mori chamber decomposition. As in the case of curves, this should yield a great deal of information about the geometry of the moduli space $\overline{\mathcal{P}}_2$.

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