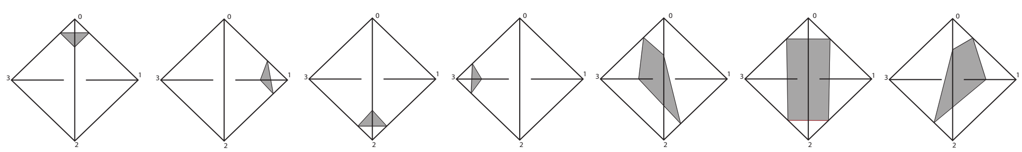
Katherine Paullin

Research Statement

Introduction

There are many questions in algorithmic topology. Given a circle embedded in $\mathbb{R}^3$, is it knotted? If we have two topological spaces, are they the same? Algorithms are a list of procedures used to solve a problem. These algorithms focus specifically on the decidability of a problem. Problems in 2-dimensions are easily solved, where as algorithmic questions in 5 or more dimensions are intractable. Thus, this leaves dimensions 3 and 4 that are the most interesting to study. My research lies in 3-manifold algorithms. A nice property of manifolds is that locally, a manifold appears like Euclidean space and for 3-manifolds specifically, many properties are determined by the surfaces they contain, which is helpful in the writing of algorithms.

The common framework for many 3-manifold algorithms is enumerating a list of surfaces and checking each item on the list for a certain characteristic. To do this, we first represent a manifold, M, as a triangulation. Kneser [4] first introduced the concept of a normal surface; normal surfaces are ones that intersect each tetrahedron of a triangulation like a plane and a tetrahedron would intersect in Euclidean 3-space. These intersections are called elementary disk types and are pictured below. Also, normal surfaces can be completely described by the number of each disk type they contain, which gives us a vector representation for each normal surface. Haken [2] later showed that every incompressible surface is isotopic to a normal surface, thus allowing for a finite representation for these surfaces.



Haken expanded the work on normal surface theory to provide an algorithmic basis for which other surfaces can be described. His work allowed us to write any surface in terms of fundamental surfaces. Since every normal surface can be written as a sum of fundamental surfaces, then Haken could use fundamentals to enumerate a list of surfaces and check them for a given property. Jaco and Oertel [3] then wrote an algorithm using these techniques to determine if a given 3-manifold is a Haken manifold, a specific type of compact irreducible 3-manifold.

Rubinstein [5] and Thompson [8] described an algorithm to recognize the 3-sphere, a groundbreaking discovery that introduced almost normal surfaces, surfaces that are normal everywhere except in one tetrahedron. Stocking [7] built on Rubinstein's [6] work and proved that any strongly irreducible splitting of a irreducible manifold is isotopic to an almost normal surface.

The next logical question to consider is the recognition of a Heegard Splitting. The simplest case in this recognition problem is the Lens Space, which contains a splitting torus. It is worth noting that tori tend to be a problem with the enumeration framework for the previous algorithms. Hence, a new algorithmic framework is necessary.

Walsh [9] proved that in a hyperbolic manifold any incompressible surface that is not a virtual fiber can be isotoped into spun normal form. A spun normal surface is a surface that is normal everywhere in the triangulation but is allowed to be infinite in the neighborhood of a vertex. Even given this infinite structure, a sequence of normalizations may still terminate. Walsh proved under what conditions a surface may be spun normalized. This leads to the question of "what surfaces can be spun almost normalized?", the main result of my dissertation work.

My work has focused on how surfaces intersect with tori, generalizing the work of Walsh to apply to arbitrary 3-manifolds, and generalizing Rubinstien and Stocking's work to extend the idea to spun almost normalization. Having spun normalization and spun almost normalization is a first step towards being able to recognize Heegard Splittings in our list of surfaces which provides a new framework for enumeration.

Spun Normalization and Spun Almost Normalization

Our input surfaces are assumed to be compact so that they intersect our triangulation in a finite number of components. In Walsh's work [9] on spun normalization, she extended a triangulated surface into a neighborhood of a vertex, and then added infinitely many copies of that piece of the surface into the neighborhood. This allows us to represent a surface with boundary as a noncompact surface. After doing so, she normalized the spun surface. Although the surface is infinite in coordinates relating to particular tori, the normalization did terminate. The first major result of my dissertation was to generalize Walsh's work to arbitrary 3-manifolds. This resulted in the following theorem:

Theorem 1. *If M is a closed orientable irreducible triangulated 3-manifold, $F \subset M$ is a disjoint union of normal tori, $X = M - nbhd(F)$, and $S \subset X$ is incompressible and ∂-incompressible then S is isotopic to a spun normal surface.*

My second main result in research has been to apply the concepts of spun normalization to almost normal surfaces. Bachman, Derby-Talbot, and Sedgwick [1] have shown that a strongly irreducible surface can be isotoped to be almost normal. By compressing along compressing disks of different types, I can reduce the complexity of a compression sequence. My dissertation work extends this compression sequence to spun almost normal surfaces.

Theorem 2. *If M is a closed orientable irreducible triangulated 3-manifold, $F \subset M$ is a disjoint union of normal tori, $X = M - nbhd(F)$, and $S \subset X$ is strongly irreducible and ∂-strongly irreducible then S can be spun almost normalized.*

Future Work

The results above are the first step in an algorithmic program to study Heegard Splittings. It is hoped that this will lead to the development of an algorithm to check for Lens Space recognition, determining if a given topological space is a Lens space. Lens spaces have Heegard genus 1, hence the simplest of the Heegard Splitting cases. I plan to investigate further techniques which would allow enumeration of Heegard Splittings and allow for us to check the Heegard genus of a list of enumerated surfaces.

As stated in my teaching statement, I hope to pursue some research in the field of Mathematics Education. I'm currently following how Common Core standards and state curricula are affecting the pedigree of our incoming freshman students. With the influx of students in need of developmental math courses, I find it interesting and important to keep up on best teaching practices and I hope to someday contribute to this field of research.

Works Cited

[1] David Bachman, Ryan Derby-Talbot, and Eric Sedgwick. Almost normal surfaces with boundary. *ArXiv e-prints*, March 2012.

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