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Research Summary and Agenda

My research interests lie in analysis and partial differential equations: currently, I am interested in the theory of quantitative homogenization of elliptic equations and its consequences. In this research statement, I will provide some background to the field and explain in some detail a few of my major results.

1 Introduction to Homogenization

1.1 Background. Composite materials such as concrete or plywood consist of multiple constituents with different physical or chemical attributes (e.g., conductivity, elastic modulus) fused together in some matrix material and well-mixed. The well-mixed material generally retains "better" characteristics than any of the constituents individually and can essentially be described as a homogeneous material, even though at the microscopic level the material maintains its heterogeneity. The theory of homogenization seeks to model composite materials as effectively homogeneous in spite of its complex heterogeneity.

Various composite materials are modeled by second-order divergence form elliptic systems with rapidly oscillating coefficients with some assumed self-repeating structure. For the purpose of this document, we will consider the case of linear elasticity. That is, we consider the system

$$\begin{cases} \mathcal{L}_\varepsilon(u_\varepsilon) := -\text{div}\,(A^\varepsilon \nabla u_\varepsilon) = F & \text{in } \Omega \subset \mathbb{R}^d \\ u_\varepsilon = f & \text{on } \partial\Omega \end{cases} \tag{1}$$

where $u_\varepsilon : \Omega \to \mathbb{R}^d$ represents the displacement of a composite material Ω subject to some body force F, f denotes a prescribed deformation along the boundary of Ω, and $A^\varepsilon = A(\cdot/\varepsilon)$ encodes at the small-scale—denoted by ε—parameters that depend on the constituents. For the simplicity of presentation, we will assume the coefficients A are periodic satisfying

$$A(\xi + z) = A(\xi) \quad \text{for } \xi \in \mathbb{R}^d,\, z \in \mathbb{Z}^d, \tag{2}$$

even though results exist for the case when A is either almost-periodic or random [3, 2, 23]. With a physically appropriate boundary condition and the assumption that $A(\cdot/\varepsilon) = \{a_{ij}^{\alpha\beta}(\cdot/\varepsilon)\}$ is real, measurable, and satisfies the elasticity conditions

$$a_{ij}^{\alpha\beta}(\xi) = a_{ji}^{\beta\alpha}(\xi) = a_{\alpha j}^{i\beta}(\xi) \tag{3}$$

$$\kappa_1|m|^2 \le a_{ij}^{\alpha\beta}(\xi) m_i^\alpha m_j^\beta \le \kappa_2 |m|^2 \tag{4}$$

for $\xi \in \mathbb{R}^d$, $0 < \kappa_1 \le \kappa_2$, and symmetric $m = \{m_i^\alpha\} \in \mathbb{R}^{d\times d}$, a solution to (1) is known to exist. As ε tends to 0, the composite material in theory becomes more homogeneous.

The following is a classical result in homogenization of elliptic equations with periodic coefficients [26, 16, 17, 1, 8, 9, 19].

Theorem 1. *Suppose A satisfies* (2), (3), *and* (4). *If $F \in L^2(\Omega;\mathbb{R}^d)$, $f \in H^{1/2}(\partial\Omega;\mathbb{R}^d)$, and u_ε solves* (1)*, then there exists a $u_0 \in H^1(\Omega;\mathbb{R}^d)$ with $u_\varepsilon \rightharpoonup u_0$ weakly in $H^1(\Omega;\mathbb{R}^d)$ and consequently $u_\varepsilon \to u_0$ strongly in $L^2(\Omega;\mathbb{R}^d)$. Moreover, u_0 satisfies*

$$\begin{cases} \mathcal{L}_0(u_0) = F & \text{in } \Omega, \\ u_0 = f & \text{on } \partial\Omega \end{cases} \tag{5}$$

where $\mathcal{L}_0 = -div(\widehat{A}\nabla)$ and $\widehat{A}$ are constant coefficients.

The constant coefficients $\widehat{A}$ referenced above are an effective "average" of the characteristics in the microscopic constituents of the composite material [8].

1.2 Quantitative Convergence. Recently, there has been a growing interest in quantitative results regarding the convergence of solutions u_ε [6, 7, 4, 22, 19]. Indeed, as $\varepsilon \to 0$, computing numerical solutions to (1) becomes unwieldy, whereas (5) can be solved efficiently. Even though the limiting case is physically unrealistic, a quantitative analog to Theorem 1 would numerically justify solving (5) instead of (1) for ε small. The following theorem should be considered as an analog to Theorem 1.

Theorem 2. *Suppose A satisfies (2), (3), and (4). Let $\chi_j^\beta \in W^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d)$ satisfy the following boundary value problem*

$$\begin{cases} \mathcal{L}_1(\chi_j^\beta + P_j^\beta) = 0 \quad \text{in } Q := [0,1)^d \\ \displaystyle\int_Q \chi_j^\beta = 0, \ \ \chi_j^\beta \text{ is 1-periodic} \end{cases} \tag{6}$$

for $1 \leq j, \beta \leq d$, where $P_j^\beta(\xi) = \xi_j e^\beta$. If u_ε, u_0 solve (1), (5), respectively, then there exists a constant C independent of ε such that

$$\|u_\varepsilon - u_0 - \varepsilon\chi_j^\beta(\cdot/\varepsilon)(\partial u_0^\beta/\partial x_j)\|_{H^1(\Omega)} \leq C\varepsilon^{1/2}\|u_0\|_{H^3(\Omega)}.$$

The 1-periodic functions $\chi = \{\chi_j^\beta\}$ satisfying (6) are referred to as the first-order correctors corresponding to the system (1). In general, if A only satisfies (2), (3), and (4), then neither the first-order correctors nor the gradient of the first-order correctors is *a priori* locally bounded. This assumption has since been removed through the use of a smoothing operator (see Theorem 5), which was inspired by work in [25, 14, 24]. The effective coefficients $\widehat{A} = \{\widehat{a}_{ij}^{\alpha\beta}\}$ are given by

$$\widehat{a}_{ij}^{\alpha\beta} = \fint_Q a_{ik}^{\alpha\gamma}\frac{\partial}{\partial y_k}\left(\chi_j^{\gamma\beta} + P_j^\beta\right).$$

It is know that if A satisfies (3) and (4), then $\widehat{A}$ satisfies (3) and (4) with the same constants κ_1, κ_2.

1.3 Regularity. When given a boundary value problem such as (1), a natural concern is regularity of solutions: if F and f retain some smoothness in Ω and on $\partial\Omega$, respectively, how is the regularity of u_ε affected? Indeed, *a priori* Lipschitz estimates uniform in ε are best possible since ∇u_ε converges only weakly [5]. Optimal regularity estimates have been derived from the qualitative convergence of Theorem 1 (using the so-called *method of compactness* [5, 27, 29, 14]) and also from the quantitative convergence of Theorem 2 [3, 22, 23, 18].

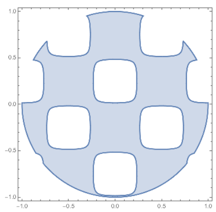
Theorem 3. *Suppose A satisfies (2), (3), and (4), and suppose Ω is a $C^{1,\alpha}$ domain. Let $F \in L^p(\Omega)$ for some $p > d$ and $f \in C^{1,\alpha}(\partial\Omega)$ for an $\alpha \in (0,1)$. If u_ε solves (1), then there exists a $C = C(\kappa_1, \kappa_2, \Omega, d, p, \omega)$ such that*

$$[u_\varepsilon]_{C^{0,1}(\Omega)} \leq C\left\{[f]_{C^{1,\alpha}(\partial\Omega)} + \|F\|_{L^p(\Omega)}\right\}.$$

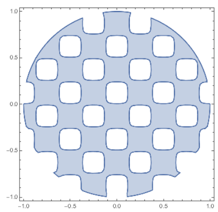
2 Current Research

In my current research, I consider composite materials reinforced at the miscroscopic level with *perforations* or *soft inclusions* (e.g., electrical or thermal insulators, constituents with low elastic modulus). Soft inclusions are substantially "weaker" than the surrounding matrix material. Typically, embedded inclusions or perforations reduce material cost and alter performance. For example, in lightweight aggregate concrete, compressive strength decreases with increasing volume of inclusions, but inclusions can increase thermal inertia and improve energy efficiency [13].

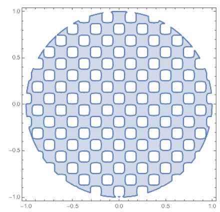
Let $\omega \subset \mathbb{R}^d$ denote an unbounded (possibly anisotropic) material substrate and $\mathbb{R}^d \backslash \omega$ denote the placement of perforations or inclusions at the microscopic scale. For example, if $d = 2$, $\Omega = B(0,1)$, and $\omega = \{(x_1, x_2) : \cos(2\pi x_1)\sin(2\pi x_2) < 0.1\}$, then for various values of ε the set $\Omega_\varepsilon := \Omega \cap \varepsilon\omega$ looks as follows.



Ω_1



$\Omega_{0.5}$



$\Omega_{0.25}$

In particular, as $\varepsilon \to 0$, the size of each inclusion or perforation decreases, but the number increases. We are concerned with the weakly-formulated boundary value problem

$$\begin{cases} \displaystyle\int_\Omega k^\varepsilon_{\delta^2} A^\varepsilon \nabla u_{\varepsilon,\delta} \cdot \nabla \varphi = 0, \quad \text{for any } \varphi \in H^1_0(\Omega; \mathbb{R}^d) \\ u_{\varepsilon,\delta} - f \in H^1_0(\Omega; \mathbb{R}^d), \end{cases} \tag{7}$$

where

$$k_{\delta^2}(x/\varepsilon) = \mathbf{1}_+(x/\varepsilon) + \delta^2 \mathbf{1}_-(x/\varepsilon),$$

the functions $\mathbf{1}_+$ and $\mathbf{1}_-$ denote characteristic functions of ω and $\mathbb{R}^d \backslash \omega$, respectively, and the modulus of $\delta \in [0,1]$ determines the nature of the prescribed inclusions.

Note if $\delta = 1$, then (7) corresponds with the weak formulation of (1) with $F = 0$. The case when $\delta = 0$ is typically referred to as homogenization in perforated domains [10, 15, 19, 21], while the case $\delta > 0$ but small is typically referred to as homogenization with soft inclusions [19]. The challenge of (7) is that the coefficients $k_{\delta^2} A$ do not satisfy (4) uniformly in $\mathbb{R}^d$. A solution to (4) is known to exist [15, 19], although it is not bounded uniformly in δ in $H^1(\Omega; \mathbb{R}^d)$.

2.1 Homogenization in perforated domains. When $\delta = 0$, equation (7) is the weak formulation of the mixed boundary value problem

$$\begin{cases} \mathcal{L}_\varepsilon(u_\varepsilon) = 0 \quad \text{in } \Omega_\varepsilon := \Omega \cap \varepsilon\omega \\ -n_\varepsilon A^\varepsilon \nabla u_\varepsilon = 0 \quad \text{on } S_\varepsilon := \partial\Omega_\varepsilon \cap \Omega \\ u_\varepsilon = f \quad \text{on } \Gamma_\varepsilon := \partial\Omega_\varepsilon \cap \partial\Omega, \end{cases} \tag{8}$$

where I have suppressed the subscript $\delta = 0$ for simplicity of notation, n_ε denotes the exterior unit vector normal to S_ε, and the coefficients A are assumed to satisfy (2), (3), and (4) but only in the connected substrate ω. In particular, we look for solutions $u_\varepsilon \in H^1(\Omega_\varepsilon, \Gamma_\varepsilon; \mathbb{R}^d)$, where $H^1(\Omega_\varepsilon, \Gamma_\varepsilon; \mathbb{R}^d)$ denotes the closure in $H^1(\Omega_\varepsilon; \mathbb{R}^d)$ of $C^\infty(\mathbb{R}^d; \mathbb{R}^d)$ functions vanishing on Γ_ε. Define the first-order correctors $\chi = \{\chi^\beta_j\}$ for (8) by the mixed boundary value problem

$$\begin{cases} \mathcal{L}_1\left(\chi^\beta_j + P^\beta_j\right) = 0 \quad \text{in } Q \cap \omega \\ -nA\nabla\left(\chi^\beta_j + P^\beta_j\right) = 0 \quad \text{on } Q \cap \partial\omega \\ \chi^\beta_j \text{ is 1-periodic}, \ \displaystyle\int_{Q\cap\omega} \chi^\beta_j = 0. \end{cases}$$

Let

$$r_\varepsilon = u_\varepsilon - u_0 - \varepsilon\chi\left(\cdot/\varepsilon\right) K_\varepsilon\left((\nabla u_0)\eta_\varepsilon\right), \tag{9}$$

where K_ε is a smoothing operator at the scale ε, u_0 is a solution of the homogenized problem (5) with constant coefficients $\widehat{A} = \{\widehat{a}_{ij}^{\alpha\beta}\}$ defined by

$$\widehat{a}_{ij}^{\alpha\beta} = \fint_{Q\cap\omega} a_{ij}^{\alpha\beta} \frac{\partial}{\partial \xi_j} \left(\chi_j^\beta + P_j^\beta\right) d\xi$$

In my work [21], I proved the following theorem, which is an improvement on known results: I have removed the regularity assumption on the correctors and lowered the regularity required of the homogenized solution. In particular, I establish the optimal H^1-convergence rate for solutions to (8) in the connected domain $\varepsilon\omega$ under minimal assumptions.

Theorem 4. *Let Ω be a bounded Lipschitz domain and ω denote an unbounded Lipschitz domain with 1-periodic structure, i.e., the characteristic function $\mathbf{1}_+$ of ω satisfies* (2). *Suppose A satisfies* (2), (3), *and* (4). *Let u_ε solve* (8). *There exists a constant $C = C(\kappa_1, \kappa_2, \Omega, \omega, d)$ such that*

$$\|\nabla r_\varepsilon\|_{L^2(\Omega_\varepsilon)} \le C\varepsilon^{1/2}\|f\|_{H^1(\partial\Omega)}, \tag{10}$$

where r_ε is given by (9).

The argument relies on the energy estimates of the boundary value problem

$$\begin{cases} \mathcal{L}_\varepsilon(r_\varepsilon) = -\mathcal{L}_\varepsilon(u_0) - \mathcal{L}_\varepsilon(\varepsilon\chi^\varepsilon K_\varepsilon^2((\nabla u_0)\eta_\varepsilon)) & \text{in } \Omega_\varepsilon, \\ -n_\varepsilon A^\varepsilon \nabla r_\varepsilon = -n_\varepsilon A^\varepsilon \nabla u_0 - n_\varepsilon A^\varepsilon \nabla(\varepsilon\chi^\varepsilon K_\varepsilon^2((\nabla u_0)\eta_\varepsilon)) & \text{on } S_\varepsilon, \\ r_\varepsilon = 0 \quad \text{on } \Gamma_\varepsilon \end{cases}$$

and nontangential maximal function estimates from the work of Dahlberg, Verchota, and Kenig for solutions to constant coefficient elliptic equations in Lipschitz domains [11]. Indeed, with these estimates I show

$$\left|\int_{\Omega_\varepsilon} A^\varepsilon \nabla r_\varepsilon \cdot \nabla w\right| \le C\varepsilon^{1/2}\|f\|_{H^1(\partial\Omega)}\|\nabla w\|_{L^2(\Omega_\varepsilon)}.$$

To quantify the difference between the geometries of (8) and (5), I rely on an extension operator P_ε [19]. In particular, if Ω_0 is a bounded Lipschitz domains with $\overline{\Omega} \subset \Omega_0$, $\operatorname{dist}(\overline{\Omega}, \partial\Omega_0) > 1$, then for $\varepsilon < 1$ there exists a linear extension operator $P_\varepsilon : H^1(\Omega_\varepsilon, \Gamma_\varepsilon; \mathbb{R}^d) \to H_0^1(\Omega; \mathbb{R}^d)$ with

$$\|\nabla P_\varepsilon w\|_{L^2(\Omega_0)} \le C\|\nabla w\|_{L^2(\Omega_\varepsilon)}.$$

With $P_\varepsilon(w)\eta_\varepsilon \in H^1(\Omega, \Gamma_\varepsilon; \mathbb{R}^d)$, I relate the geometries: for $w \in H^1(\Omega_\varepsilon, \Gamma_\varepsilon; \mathbb{R}^d)$,

$$\int_{\Omega_\varepsilon} A^\varepsilon \nabla u_\varepsilon \cdot \nabla w = \int_\Omega \widehat{A}\nabla u_0 \cdot \nabla\left[P_\varepsilon(w)\eta_\varepsilon\right].$$

To counter the growth of $\nabla\eta_\varepsilon$ as $\varepsilon \to 0$, I establish a Poincaré-type inequality for functions defined in Ω and vanishing only on Γ_ε.

2.2 Homogenization with soft inclusions. To extend my work with perforated domains, I subsequently considered homogenization with soft inclusions, i.e., the weakly formulated boundary value problem (7) with $\delta \in (0, 1]$. To establish an H^1-convergence rate, we first define the first-order correctors $\chi_\delta = \{\chi_{j,\delta}^\beta\}$ for the system (7) by

$$\begin{cases} \displaystyle\int_Q k_{\delta^2} a_{ik}^{\alpha\gamma} \frac{\partial}{\partial \xi_k}\left(\chi_{j,\delta}^{\gamma\beta} + P_j^{\gamma\beta}\right) \frac{\partial \varphi^\alpha}{\partial \xi_i}\, d\xi = 0 \quad \text{for any } \varphi \in H_{\text{per}}^1(Q; \mathbb{R}^d) \\ \chi_{j,\delta}^\beta \text{ is 1-periodic}, \quad \displaystyle\int_Q \chi_{j,\delta}^\beta = 0. \end{cases} \tag{11}$$

Similar to the case $\delta = 0$, let $u_{0,\delta}$ denote the weak solution to the boundary value problem (5), where the constant matrix $\widehat{A}_\delta = \{\widehat{a}_{ij,\delta}^{\alpha\beta}\}_{i,j,\alpha,\beta}$ is defined by

$$\widehat{a}_{ij,\delta}^{\alpha\beta} = \fint_Q k_\delta a_{ik}^{\alpha\gamma} \frac{\partial}{\partial \xi_k} \left(\chi_{j,\delta}^{\gamma\beta} + P_j^{\gamma\beta} \right).$$

My work in [20]—which is in preparation—provides a convergence rate for $u_{\varepsilon,\delta}$ in $H^1(\Omega;\mathbb{R}^d)$. Again, no regularity assumptions are made on the coefficients or correctors, unlike in Theorem 2. Also, it is important to emphasize the constant C in the following theorem is completely independent of ε and δ. Let

$$r_{\varepsilon,\delta} = u_{\varepsilon,\delta} - u_{0,\delta} - \varepsilon\chi_\delta(\cdot/\varepsilon)K_\varepsilon^2((\nabla u_{0,\delta})\eta_\varepsilon). \tag{12}$$

Theorem 5. *Let Ω be a bounded Lipschitz domain and ω denote an unbounded Lipschitz domain with 1-periodic structure, i.e., the characteristic function $\mathbf{1}_+$ of ω satisfies* (2)*. Suppose A satisfies* (2)*,* (3)*, and* (4)*. Let $u_{\varepsilon,\delta}$ solve* (7) *for some $\delta \in [0,1]$. There exists a constant $C = C(\kappa_1, \kappa_2, \Omega, \omega, d)$ such that*

$$\|k_\delta^\varepsilon r_{\varepsilon,\delta}\|_{L^2(\Omega)} + \|k_\delta^\varepsilon \nabla r_{\varepsilon,\delta}\|_{L^2(\Omega)} \le C\varepsilon^\mu \|f\|_{H^1(\partial\Omega)}, \tag{13}$$

where $r_{\varepsilon,\delta}$ is given by (12) *and $\mu > 0$ is independent of ε and δ.*

The rate in (13) is indeed suboptimal. In fact, for each fixed $\delta_0 > 0$, the coefficients $k_{\delta_0^2}A$ are uniformly elliptic in $\mathbb{R}^d$, and it is known that the solution u_{ε,δ_0} converges weakly with rate $\mathcal{O}(\varepsilon^{1/2})$. The novelty of Theorem 5, however, is that the right-hand side of estimate (13) is completely independent of both ε and δ. As we will later see, any $\mu > 0$ is sufficient for establishing large-scale Lipschitz estimates.

Similar to the case $\delta = 0$, the difficulty in proving Theorem 5 lies in preserving the delicate geometry of the problem. While the domain of problem (7) is "weighted," the structure of the homogenized material and the domain of problem (5) is not. At the same time, admissible test functions in the weak formulation of (7) and (5) are expected to belong to the same space. In my soon to be submitted work, I extract information regarding the convergence in the connected substrate by reducing this problem to the same issue faced when considering perforated domains. Essentially, I show functions belonging to $H^1(\Omega, \Gamma_\varepsilon; \mathbb{R}^d)$ are "good enough" test functions in the weak formulation of (7), i.e.,

$$\left| \int_\Omega k_{\delta^2}^\varepsilon A^\varepsilon \nabla u_{\varepsilon,\delta} \cdot \nabla w \right| \le C\varepsilon^{1/2} \|f\|_{H^1(\partial\Omega)} \|\mathbf{1}_+^\varepsilon \nabla w\|_{L^2(\Omega)}.$$

To show this, I first prove a single reverse Hölder inequality for the solution $u_{\varepsilon,\delta}$ and take advantage of this higher integrability in a boundary layer of thickness ε. Then, there exists a $p > 2$ so that

$$\left(\int_{\{x\,:\,\rho(x) \le \varepsilon\}} |k_\delta \nabla u_{\varepsilon,\delta}|^2 \right)^{1/2} \le \varepsilon^{\mu_1} \left(\int_\Omega |k_\delta \nabla u_{\varepsilon,\delta}|^p \right)^{1/p} \le C\varepsilon^\mu \|f\|_{H^1(\partial\Omega)},$$

where $\rho(x) = \operatorname{dist}(x, \partial\Omega)$. The exponent $\mu = \min\{\mu_1, 2d/(d-1)\}$ gives rise to the rate in Theorem 5. It should be noted that I have not provided a $W^{1,p}$-estimate for all p. This will require large-scale Lipschitz estimates and local, microscopic $W^{1,p}$-estimates. This is listed as a direction for future work.

2.3 Regularity. The indirect method of compactness originating with Avellaneda and Lin [5] is quite complicated, especially when one considers a periodic substrate $\omega \subsetneq \mathbb{R}^d$. Indeed, the method of compactness is essentially "proof by contradiction," and the method relies on sequences of operators $\{\mathcal{L}_{\varepsilon_k}^k\}_k$ and sequences of functions $\{u_k\}_k$ satisfying $\mathcal{L}_{\varepsilon_k}^k(u_k) = 0$, where $\mathcal{L}_{\varepsilon_k}^k = -\operatorname{div}(A_k^{\varepsilon_k}\nabla)$ and $\{A_k^{\varepsilon_k}\}_k$ satisfies (2), (3), and (4). One should also consider affine transformations of the substrate, as the class of coefficients $\{A_k\}$ considered should satisfy (4) in $\omega + s_k$ for any $s_k \in \mathbb{R}^d$. One of my major research accomplishments is modifying the direct method of [4, 3] to obtain optimal regularity estimates for elliptic systems with the weak formulation (7) when $0 \leq \delta \leq 1$. The compactness method was applied by Yeh in [27, 29] to claim uniform Hölder estimates estimates in the connected substrate for equations with $\delta = \varepsilon$ and diagonal coefficients. Some of his consequential estimates do not extend to elliptic systems.

I proved the following result when $\delta = 0$ in [21], and it concerns large-scale interior Lipschitz estimates for (8). The work providing interior Lipschitz estimates in the case $\delta \in (0, 1]$ is currently in preparation. It should be noted that no smoothness assumptions are required on the coefficients or the domain ω for the following result.

Theorem 6. *Suppose A satisfies* (2), (3), *and* (4), *and suppose $u_{\varepsilon,\delta}$ solves* (7) *in some ball $B(x_0, R) \subset \mathbb{R}^d$. There exists a constant $C = C(\kappa_1, \kappa_2, d, \omega)$ so that*

$$\left(\fint_{B(x_0,r)} |k_\delta^\varepsilon \nabla u_{\varepsilon,\delta}|^2 \right) \leq C \left(\fint_{B(x_0,R)} |k_\delta^\varepsilon \nabla u_{\varepsilon,\delta}|^2 \right)^{1/2} \tag{14}$$

for any $\varepsilon \leq r \leq R$.

Indeed, if estimate (14) held for all $0 < r < R/2$, then we would be able to bound

$$\|k_\delta^\varepsilon \nabla u_{\varepsilon,\delta}\|_{L^\infty(B(x_0,R/2))} \leq C \left(\fint_{B(x_0,R)} |k_\delta^\varepsilon \nabla u_{\varepsilon,\delta}|^2 \right)^{1/2}, \tag{15}$$

which is essentially the full interior Lipschitz estimate for (8). Hence, the scale-invariant estimate of Theorem 6 should be considered as the large-scale Lipschitz estimate.

I prove estimate (14) through a Campanato-type iteration originating with Smart and Armstrong [4]. Essentially, the scheme relies on the fact that functions which can be well-approximated by $C^{1,\alpha}$ functions at multiple scales must be Lipschitz at least on scales larger enough that the approximation is valid. Hence, rates (10) and (13) are needed. If $H_{\varepsilon,\delta}$ defined by

$$H_{\varepsilon,\delta}(r) = \inf_{\substack{M \in \mathbb{R}^{d\times d} \\ q \in \mathbb{R}^d}} \left(\fint_{B(0,r)} |k_\delta^\varepsilon (u_{\varepsilon,\delta} - Mx - q)|^2 \right)^{1/2}$$

measures the "flatness" of the solution $u_{\varepsilon,\delta}$ in the connected substrate, then the convergence rate establishes a mesoscopic scale on which $u_{\varepsilon,\delta}$ is "flatter," a property it inherits from the homogenized solution $u_{0,\delta}$: there exists $\theta \in (0, 1/2)$ so that if $u_{\varepsilon,\delta}$ satisfies (7) in some ball $B(0, 1)$, then

$$H_{\varepsilon,\delta}(\theta r) \leq \frac{1}{2} H_{\varepsilon,\delta}(r) + C \left(\frac{\varepsilon}{r}\right)^{\mu(\delta)} \left(\fint_{B(0,2r)} |u_{\varepsilon,\delta}|^2 \right)^{1/2}, \quad r \in [\varepsilon, 1/2], \tag{16}$$

where $\mu(0) = 1/2$ and $\mu(\delta) \geq \delta_0 > 0$ for $\delta \in (0, 1]$. This is enough to carry out the general scheme for deriving large-scale Lipschitz estimates in homogenization that was later adapted by Armstrong and Shen [3, 22]. The adaption by Shen verifies that a Dini rate in (16) is sufficient.

Another result of my work in [21] and a consequence of Theorem 6 is a Liouville-type property for systems of linear elasticity in unbounded periodically perforated domains. In particular, if $u_{1,0}$ satisfies (7) with $\varepsilon = 1$, $\delta = 0$, and satisfies the growth condition

$$\left(\fint_{B(0,R)\cap\omega} |u_{1,0}|^2 \right)^{1/2} \leq CR^\nu \tag{17}$$

for some $\nu \in (0, 1)$, $C = C(u_{1,0})$, and any $R > 1$, then $u_{1,0}$ is in fact constant.

In the case of elastic systems, estimate (15) does not *a priori* hold without additional smoothness assumptions on the coefficients. Indeed, with bounded and measurable coefficients the solution may not even be bounded, which is contrary to the case of elliptic equations where De Giorgi-Nash estimates hold. When $0 < \delta < 1$, the smoothness assumptions, the scaling $x \mapsto \varepsilon x$, and a layer potential arument for interface problems originating with Escauriaza, Fabes, and Verchot provides the full estimate [12, 27, 28]. When $\delta = 0$, I proved estimate (15) relying on further smoothness assumptions on the coefficients and scaling [21].

3 Directions for Future Work

There are still many questions to be answered with regards to system (7). Specifically, I would like to generalize more results that are more or less known for the case $\delta = 1$ or the case of constant coefficients equations. For example, boundary Lipschitz estimates uniform in ε for $\delta = 1$ have been established [22].

Question 7. *Can uniform Lipschitz estimates for* (7) *be established in* $C^{1,\alpha}$ *domains? In particular, what is the correct setting for boundary Lipschitz estimates?*

At the large-scale, with a few slight modifications boundary estimates should be clear. However, it is unclear that for systems an estimate such as (8) holds at every scale. For the case $\delta = 0$, a simple scaling gives rise to a system with mixed boundary values. Without further assumptions, a solution is known to not even be $C^{0,\alpha}$ for all $0 < \alpha < 1$. One possibility is to consider domains of type II mentioned in [19]. Specifically, type II domains only include inclusions and perforations in the interior of Ω.

Also established for the case $\delta = 1$ are $W^{1,p}$ estimates uniform in ε provided $A \in VMO(\mathbb{R}^d)$ [22]. The same L^p gradient estimates for single constant-coefficient elliptic equations in smooth domains were established by Yeh in [28] for $0 < \delta \leq 1$. When $\delta = 1$, the result follows from interior $W^{1,p}$-estimates at the scale ε, large-scale interior Lipschitz estimates, and boundary Hölder estimates. In particular, local microscopic $W^{1,p}$ estimates together with large-scale interior Lipschitz give interior $W^{1,p}$ estimates, and interior $W^{1,p}$ estimates together with large-scale boundary Hölder estimates establish boundary $W^{1,p}$ estimates.

Question 8. *For* $p \in (2, \infty)$, *if* $\mathcal{L}_{1,\delta}(u_{1,\delta}) = 0$ *in* $B(x_0, 2)$, *does the estimate*

$$\left(\fint_{B(x_0,1)} |k_\delta \nabla u_{1,\delta}|^p \right)^{1/p} \leq C \left(\fint_{B(x_0,2)} |k_\delta \nabla u_{1,\delta}|^2 \right)^{1/2} \tag{18}$$

hold for a constant $C = C(d, \kappa_1, \kappa_2, p, [A]_{VMO})$*?*

If such an estimate were to hold, then an affirmative answer to Question 7 would provide the expected L^p-gradient estimates for (7). Of course, a duality argument would provide the estimate for $p \in (1, 2)$. It should be noted that (18) is at the scale ε, and so this is really a question for interface problems arising in elasticity, i.e., elliptic systems with discontinuous coefficients having some piecewise regularity. If the coefficients are Hölder continuous, then estimate (18) follows from a layer potential argument sue to Escauriaza, Fabes, and Verchota [12]. An estimate like (18) together with optimal boundary regularity allow one to establish Rellich type estimates, which are not known to be accessible through compactness methods.

Finally, we have only considered Dirichlet boundary conditions in variational problem (4). In the case $\delta = 1$, Avellaneda and Lin considered the Neumann problem and derived regularity results using the compactness method [5], and Shen applied the direct method to derive other results for the Neumann problem, including Rellich type estimates [22]. Of course, to apply the direct method, we require a quantitative convergence rate.

Question 9. *Suppose* $u_{\varepsilon,\delta}$ *denotes a weak solution to the Neumann problem associated with the operator* $\mathcal{L}_{\varepsilon,\delta^2} = -div(k^\varepsilon_{\delta^2} A^\varepsilon \nabla)$. *What converge rate for* $u_{\varepsilon,\delta}$ *can be established, and what regularity estimates can be derived? What is the appropriate geometric setting for the Neumann problem?*

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