Chloe Urbanski Wawrzyniak

Research Statement

ceu11@math.rutgers.edu | (574)340-1718 | `math.rutgers.edu/~ceu11`

On the Stability of the Hull(s) of an n-Sphere in $\mathbb{C}^n$

In one variable complex analysis, the unit disc is the natural domain to study, because of the Riemann Mapping Theorem. Though there is no such similarly natural domain in $\mathbb{C}^n$, we can look at holomorphic embeddings of the unit disc into $\mathbb{C}^n$ to take advantage of the useful properties it has in one dimension. If the boundary of the disc maps into a manifold, we say that disc is attached to the manifold.

In my thesis problem, we investigated whether easily obtained properties of a natural embedding of the real n-sphere S^n are stable under C^3-small perturbations. The main result is a holomorphic plateau problem: we construct a manifold whose boundary is the perturbed sphere, which is foliated by holomorphic discs attached to the perturbed sphere, and which is minimal in the sense that it is the smallest set with the property that any function which is holomorphic in a neighborhood of the perturbed sphere can be extended to that set. This question has been answered in $\mathbb{C}^2$, such as in [2], however the question is considerably more challenging, for $n \geq 3$. Our results require different tools because previous techniques rely on the sphere having only real codimension 1, so that the solution to the plateau problem produces a hypersurface. The high codimension for the case of $n \geq 3$ introduces new challenges which we address in not only our construction but also the argument of its minimality.

In addition to the immediate complications posed by the high codimension, the complex structure for the sphere in $n \geq 3$ has non-isolated singularities, called CR singularities. While $S^2 \subset \mathbb{C} \times \mathbb{R}$ in $\mathbb{C}^2$ has CR singularities only at the two poles, the singularities for $S^n \subset \mathbb{C} \times \mathbb{R}^n$ in $\mathbb{C}^n$ form an entire equator of the sphere, for $n \geq 3$. We find that under C^3-small perturbations, the local and global structure of the set of singularities remains the same. Namely, the set of singularities still forms a topological sphere and the type of the singularities (nondegenerate and elliptic) remains the same.

We then solve a Riemann-Hilbert problem, modifying a construction by Alexander in [1], to obtain an $(n-1)$-parameter family of holomorphic discs attached to the perturbed sphere, away from the set of singularities. If the perturbation is $C^{k+2,\alpha}$, the regularity of this resulting manifold is $C^{k,\alpha}$. In the case that the perturbation is real analtyic, we use techniques from nonlinear analysis to show that the construction results in a real analytic manifold. In the case that the perturbation is $\mathcal{C}^\infty$ smooth, the techniques used for the finite and real analyic cases no longer work, as C^∞ is not a Banach space. Instead, we use the techniques of multi-indices introduced by Globevnik in [5], building on the indices introduced by Forstneric in [3]. Again, because of the high codimension, we must use multi-indices rather than the more straightforward indices. These tools allow us to show that in this case, the construction results in a C^∞ smooth manifold.

We then patch this construction with small discs near the singularities which were constructed by Kënig and Webster in [7] and Huang in [6]. This gives us a complete filling of the perturbed sphere by attached holomorphic discs. This filled sphere is diffeomorphic to a real $(n+1)$-ball and the construction immediately implies that this ball is contained in the desired minimal set. Finally, we are working to show that if the perturbation is real analytic, this perturbed ball is in fact exactly the desired minimal set. The techniques used for this proof do rely on the real-analytic regularity, but we hope to use the Whitney extension theorem to extend the final result to the smooth case as well.

Weighted Bergman Spaces and the Wiegerinck Problem

For a domain $\Omega \subseteq \mathbb{C}^n$, the Bergman space $A^2(\Omega)$ is the space of all holomorphic functions on Ω which are Lebesgue square-integrable there. One can easily see that if Ω is bounded, then $A^2(\Omega)$ is infinite-dimensional, since all of the holomorphic polynomials are there. It is also an easy exercise to see that $A^2(\mathbb{C}^n) = \{0\}$. In 1984, Wiegerinck proved in [8] that these are the only two possibilities for domains in $\mathbb{C}$. In the same paper, he showed that this dichotomy does not immediately extend to higher dimensions by constructing non-pseudoconvex domains in $\mathbb{C}^n$, $n \geq 2$, for any possible finite dimension. However, this left a real gap in our understanding of this question. Pseudoconvex domains are in a sense the natural domains on which to expect one-dimensional phenomena to occur, as all domains in $\mathbb{C}$ are pseudoconvex. This gap led to a natural question which has since been referred to as the Wiegerinck problem: does there exist a domain $\Omega \subset \mathbb{C}^n$ for $n \geq 2$ which is pseudoconvex such that $A^2(\Omega)$ is finite-dimensional and nontrivial?

Pieces of this problem have been chipped away at. For example, in 2017 Gallagher, Harz, and Herbert proved in [4] that if $\partial\Omega$ has a point of strict pseudoconvexity, then $A^2(\Omega)$ is infinite dimensional.

At a recent invited workshop at the American Institute of Mathematics (AIM), I worked with a group of other mathematicians to investigate the dimension of the weighted Bergman space, i.e. the space of holomorphic functions which are square-integrable with respect to a weight function. Little work has been done in this area, and some of our preliminary results indicate that this may lead to more information about the solution to the Weigerinck problem. For example, we constructed a sequence of smooth, subharmonic functions ϕ_k such that $\dim A^2(\mathbb{C}, e^{-\phi_k}) = k$, and we believe the construction can be generalized to higher dimensions.

As the AIM organizers intended, we left the workshop with more questions than answers, and I believe this will be a fruitful question for further investigation. Outside of the interest in the question of weighted Bergman spaces itself, we believe this line of questioning may lead to a solution to the unweighted Wiegerinck problem.

References

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