CLASS NOTES
MATH 551 (FALL 2014)

BERTRAND GUILLOU

1. Wed, Aug. 27

Topology is the study of shapes. (The Greek meaning of the word is the study of places.) What kind of shapes? Many are familiar objects: a circle or triangle or square. From the point of view of topology, these are indistinguishable. Going up in dimension, we might want to study a sphere or box or a torus. Here, the sphere *is* topologically distinct from the torus. And neither of these is considered to be equivalent to the circle.

One standard way to distinguish the circle from the sphere is to see what happens when you remove two points. One case gives you two disjoint intervals, whereas the other gives you an (open) cylinder. The two intervals are disconnected, whereas the cylinder is not. This then implies that the circle and the sphere cannot be identified as topological spaces. This will be a standard line of approach for distinguishing two spaces: find a topological property that one space has and the other does not.

In fact, all of the above examples arise as **metric spaces**, but topology is quite a bit more general. For starters, a circle of radius 1 is the same as a circle of radius 123978632 from the eyes of topology. We will also see that there are many interesting spaces that can be obtained by modifying familiar metric spaces, but the resulting spaces cannot always be given a nice metric.

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As we said, many examples that we care about are metric spaces, so we'll start by reviewing the theory of metric spaces.

Definition 1.1. A **metric space** is a pair (X, d), where X is a set and $d : X \times X \longrightarrow \mathbb{R}$ is a function (called a "metric") satisfying the following three properties:

1. (Symmetry) $d(x, y) = d(y, x)$ for all $x, y \in X$
2. (Positive-definite) $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$
3. (Triangle Inequality) $d(x, y) + d(y, z) \geq d(x, z)$ for all x, y, z in X.

Example 1.2.
1. $\mathbb{R}$ is a metric space, with $d(x, y) = |x - y|$.
2. $\mathbb{R}^2$ is a metric space, with $d(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$. This is called the standard, or Euclidean metric, on $\mathbb{R}^2$.
3. $\mathbb{R}^n$ similarly has a Euclidean metric, defined by
$$d(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}.$$
4. $\mathbb{R}^2$, with $d(\mathbf{x}, \mathbf{y}) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$.

Date: December 12, 2014.

(5) $\mathbb{R}^2$, with $d(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + |x_2 - y_2|$.

Given a point x in a metric space X, we can consider those points "near to x".

Definition 1.3. Let (X, d) be a metric space and let $x \in X$. We define the (open) **ball of radius** r **around** x to be

$$B_r(x) = \{y \in X \mid d(x, y) < r\}.$$

Example 1.4.
(1) In $\mathbb{R}$, with the usual metric, we have $B_r(x) = (x - r, x + r)$.
(2) In $\mathbb{R}^2$, with the standard metric, we have $B_r(\mathbf{x})$ is a disc of radius r, centered at $\mathbf{x}$.
(3) In $\mathbb{R}^n$, with the standard metric, we have $B_r(\mathbf{x})$ is an n-dimensional ball of radius r, centered at $\mathbf{x}$.
(4) In $\mathbb{R}^2$, with the max metric, $B_r(\mathbf{x})$ takes the form of a square, with sides of length $2r$, centered at $\mathbf{x}$.
(5) In $\mathbb{R}^2$, with the "taxicab" metric, $B_r(\mathbf{x})$ is a diamond, with sides of length $r\sqrt{2}$, centered at $\mathbf{x}$.

In the definition of a metric space, we had a metric function $X \times X \longrightarrow \mathbb{R}$. Let's review: what is the set $X \times X$? More generally, what is $X \times Y$, when X and Y are sets. We know this as the set of ordered pairs

$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}.$$

This is the usual definition of the **cartesian product** of two sets. One of the points of emphasis in this class will be not just objects or constructions but rather maps into/out of objects. With that in mind, given the cartesian product $X \times Y$, can we say anything about maps into or out of $X \times Y$?

The first thing to note is that there are two "natural" maps out of the product; namely, the projections. These are

$$p_X : X \times Y \longrightarrow X, \qquad p_X(x, y) = x$$

and

$$p_Y : X \times Y \longrightarrow Y \qquad p_Y(x, y) = y.$$

Now let's consider functions *into* $X \times Y$ from other, arbitrary, sets. Suppose that Z is a set. How would one specify a function $f : Z \longrightarrow X \times Y$? For each $z \in Z$, we would need to give a point $f(z) \in X \times Y$. This point can be described by listing its X and Y coordinates. Given that the projection p_X takes a point in the product and picks out its X-coordinate, it follows that the function f_X defined as the composition

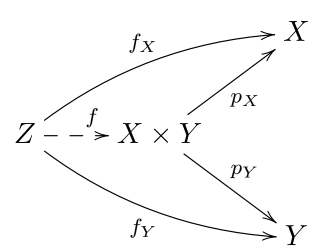
$$Z \xrightarrow{f} X \times Y \xrightarrow{p_X} X$$

is the function of X-coordinates of the function f. We similarly get a function f_Y by using p_Y instead.

And the main point of this is that the function f contains the same information as the pair of functions f_X and f_Y.

Proposition 1.5. *(Universal property of the cartesian product) Let X, Y, and Z be any sets. Suppose given functions $f_X : Z \longrightarrow X$ and $f_Y : Z \longrightarrow Y$. Then there exists a* **unique** *function $f : Z \longrightarrow X \times Y$ such that*

$$f_X = p_X \circ f, \qquad \text{and} \qquad f_Y = p_X \circ f.$$



Furthermore, it turns out that the above property uniquely characterizes the cartesian product $X \times Y$, up to bijection. We called this a "Proposition", but there is nothing difficult about this, once you understand the statement. The major advance at this point is simply the reframing of a familiar concept. We will see later in the course why this is useful.

As we already said, we will promote the viewpoint that it is not just objects that are important, but also maps. We have introduced the concept of a metric space, so we should then ask "What are maps between metric spaces"?

The strictest answer is what is known as an **isometry**: a function $f : X \longrightarrow Y$ such that $d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2)$ for all pairs of points x_1 and x_2 in X. This is a perfectly fine answer in many regards, but for our purposes, it will be too restrictive. For instance, what are all isometries $\mathbb{R} \longrightarrow \mathbb{R}$?

We will prefer to study the more general class of **continuous** functions.

2. Fri, Aug. 29

Today, we will discuss continuous functions. But first, let's return briefly to the "universal property of the product from last time". This property asserted the existence of a **unique** function f, given the coordinate functions f_X and f_Y. We talked through the existence: given the coordinate functions f_X and f_Y, we define $f(z) = (f_X(z), f_Y(z))$. But the proposition states that this is the only possible definition of f. Why? Consider some arbitrary $g : Z \longrightarrow X \times Y$ such that $p_X \circ g = f_X$ and $p_Y \circ g = f_Y$. These two equations say that the X and Y coordinates of $g(z)$ are $f_X(z)$ and $f_Y(z)$, respectively. In other words, the coordinate expression for g is the formula we wrote down for f, so $g = f$.

Again, so far, it probably seems like we're taking something relatively simple and making it sound very complicated, but I promise this point of view will pay off down the line!

Definition 2.1. A function $f : X \longrightarrow Y$ between metric spaces is **continuous** if for every $x \in X$ and for every $\varepsilon > 0$, there is a $\delta > 0$ such that whenever $x' \in B_\delta(x)$, then $f(x') \in B_\varepsilon(f(x))$.

This is the standard definition, taken straight from Calc I and written in the language of metric spaces. However, it is not always the most convenient formulation.

Proposition 2.2. *Let $f : X \longrightarrow Y$ be a function between metric spaces. The following are equivalent:*

(1) *f is continuous*
(2) *for every $x \in X$ and for every $\varepsilon > 0$, there is a $\delta > 0$ such that*
$$B_\delta(x) \subseteq f^{-1}(B_\epsilon(f(x)))$$
(3) *For every $y \in Y$ and $\epsilon > 0$ and $x \in X$, if $f(x) \in B_\epsilon(y)$, then there exists a $\delta > 0$ such that*
$$B_\delta(x) \subseteq f^{-1}(B_\epsilon(y))$$
(4) *For every $y \in Y$ and $\epsilon > 0$ and $x \in X$, if $x \in f^{-1}(B_\epsilon(y))$, then there exists a $\delta > 0$ such that*
$$B_\delta(x) \subseteq f^{-1}(B_\epsilon(y))$$

The property that $f^{-1}(B_\epsilon(y))$ satisfies in condition (4) is important, and we give it a name:

Definition 2.3. Let $U \subseteq X$ be a subset. We say that U is **open** in X if whenever $x \in U$, then there exists a $\delta > 0$ such that $B_\delta(x) \subseteq U$.

With this language at hand, we can restate condition (4) above as

(4′) For every $y \in Y$ and $\epsilon > 0$, $f^{-1}(B_\epsilon(y))$ is open in X.

The language suggests that an open ball should count as an open set, and this is indeed true.

Proposition 2.4. *Let $c \in X$ and $\epsilon > 0$. Then $B_\epsilon(c)$ is open in X.*

Proof. Suppose $x \in B_\epsilon(c)$. This means that $d(x,c) < \epsilon$. Write d for this distance. Let

$$\delta = \epsilon - d.$$

We claim that this is the desired δ. For suppose that $u \in B_\delta(x)$. Then

$$d(u,c) \leq d(u,x) + d(x,c) < \delta + d = \epsilon.$$

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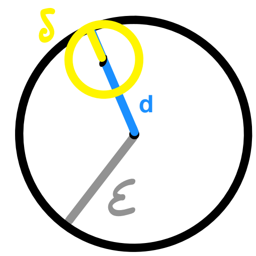

Ok, so the notion of open set is closely related to that of open ball: every open ball is an open set, and every open set is required to contain a number of these open balls. Even better, we have the following result:

Proposition 2.5. *A subset $U \subseteq X$ is open if and only if it can be expressed as a union of open balls.*

Proof. Suppose U is open, and let $x \in U$. By definition, there exists $\delta_x > 0$ with $B_{\delta_x}(x) \subseteq U$. Since this is true for every $x \in U$, we have

$$\bigcup_{x \in U} B_{\delta_x}(x) \subseteq U.$$

But every $x \in U$ is contained in the union, so clearly U must also be contained in the union. It follows that

$$\bigcup_{x \in U} B_{\delta_x}(x) = U.$$

Now suppose, on the other hand, that $U = \bigcup_\alpha B_{\delta_\alpha}(x_\alpha)$. We wish to show that U is open. Well, suppose $u \in U$. Since U is expressed as a union, this implies that $u \in B_{\delta_\alpha}(x_\alpha)$ for some α. This ball is contained in U by the definition of U, so we are done. ■

Corollary 2.6. *Any union of open subsets of X is open.*

With this description of open sets in hand, we give what is often the most useful characterization of continuous maps.

Proposition 2.7. *Let $f : X \longrightarrow Y$ be a function between metric spaces. The following are equivalent:*

(1) *f is continuous*
(5) *For every open subset $V \subseteq Y$, the preimage $f^{-1}(V)$ is open in X.*

Proof. It is clear that (5) implies (4′), which is equivalent to (1) by Prop 2.2. Now assume (1), or, equivalently, (4′). Let $V \subseteq Y$ be open. By the previous result, V is a union of balls, and by (4′) we know that the preimage of each ball is open. Using Corollary 2.6, it follows that $f^{-1}(V)$ is open. ■

For example, let's show that the translation map $t : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $t(x) = x+1$ is continuous, but that

$$f : \mathbb{R} \longrightarrow \mathbb{R}, \qquad f(x) = \begin{cases} x & x < 0 \\ x+2 & x \geq 0 \end{cases}$$

is not continuous. As we already said, a ball in $\mathbb{R}$ is an open interval, and

$$t^{-1}(a,b) = (a-1, b-1)$$

is certainly open. On the other hand, $(1,3)$ is open but $f^{-1}(1,3) = [0,1)$ is not (since it contains 0 but no ball centered at 0).

3. Wed, Sept. 3

In calculus, we are also used to thinking of continuity in terms of convergence of sequences. Recall that a sequence (x_n) in X **converges** to x if for every $\epsilon > 0$ there exists N such that for all $n > N$, we have $x_n \in B_\epsilon(x)$. We say that a "tail" of the sequence is contained in the ball around x.

Proposition 3.1. *The sequence (x_n) converges to x if and only if for every open set U containing x, some tail of (x_n) lies in U.*

Proof. Exercise. ■

Proposition 3.2. *Let $f : X \longrightarrow Y$ be a function between metric spaces. The following are equivalent:*

(1) *f is continuous*
(6) *For every convergent sequence $(x_n) \to x$ in X, the sequence $(f(x_n))$ converges to $f(x)$ in Y.*

Proof. This is on HW1. ■

This finishes our discussion of continuity.

What constructions can we make with metric spaces?

Products: Let's start with a product. That is, if (X, d_X) and (Y, d_Y) are metric spaces, is there a good notion of the product metric space? We would want to have continuous "projection" maps to each of X and Y, and we would want it to be true that to define a continuous map from some metric space Z to the product, it is enough to specify continuous maps to each of X and Y. By thinking about the case in which Z has a discrete metric, one can see that the underlying set of the product metric space would need to be the cartesian product $X \times Y$. The only question is whether or not there is a sensible metric to define.

Recall that we discussed three metrics on $\mathbb{R}^2$: the standard one, the max metric, and the taxicab metric. There, we used that $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ as an underlying set, and we combined the metrics on each copy of $\mathbb{R}$ to get a metric on $\mathbb{R}^2$. We can use the same idea here to get three different metrics on $X \times Y$, and these will all produce a metric space satisfying the right property to be a product.

For convenience, let's pick the max metric on $X \times Y$. To show that the projection $p_X : X \times Y \longrightarrow X$ is continuous, it is enough to show that each $p_X^{-1}(B_\epsilon(x))$ is open. But it is simple to show that

$$p_X^{-1}(B_\epsilon(x)) = B_\epsilon(x) \times Y$$

is open using the max metric. The same argument shows that p_Y is continuous.

Now suppose that Z is another metric space with continuous maps $f_X, f_Y : Z \rightrightarrows X, Y$. We define $f = (f_X, f_Y)$ coordinate-wise as before, and it only remains to show that it is continuous. Consider a ball $B_\epsilon(x, y) \subset X \times Y$. Under the max metric, this ball can be rewritten as

$$B_\epsilon(x, y) = B_\epsilon(x) \times B_\epsilon(y),$$

so that

$$f^{-1}(B_\epsilon(x, y)) = f^{-1}\big(B_\epsilon(x) \times B_\epsilon(y)\big) = f_X^{-1}(B_\epsilon(x)) \cap f_Y^{-1}(B_\epsilon(y)).$$

By a problem from HW1, this is open, showing that f is continuous.

Function spaces: Another important construction is that of a space of functions. That is, if X and Y are metric spaces, one can consider the set of all continuous functions $f : X \longrightarrow Y$. Is there a good way to think of this as a metric space? For example, as a set $\mathbb{R}^2$ is the same as the collection of functions $\{1, 2\} \longrightarrow \mathbb{R}$. More generally, we could consider functions $\{1, \ldots, n\} \longrightarrow Y$ or even $\mathbb{N} \longrightarrow Y$ (i.e. sequences).

Of the metrics we discussed on $\mathbb{R}^2$, the max metric generalizes most easily to give a metric on $Y^\infty = Y^\mathbb{N}$. We provisionally define the **sup metric** on the set of sequences in Y by

$$d_{\text{sup}}((y_n),(z_n)) = \sup_n \{d_Y(y_n,z_n)\}.$$

Without any further restrictions, there is no reason that this supremum should always exist. If Y is a bounded metric space, or if we only consider bounded sequences, then we are OK. Another option is to arbitrarily truncate the metric.

Lemma 3.3. *Let (Y,d) be a metric space. Define the resulting bounded metric $\overline{d}$ on Y by*

$$\overline{d}(y,z) = \min\{d(y,z),1\}.$$

This is a metric, and the open sets determined by $\overline{d}$ are precisely the open sets determined by d.

We now redefine the sup metric on Y^∞ to be

$$d_{\text{sup}}((y_n),(z_n)) = \sup_n \{\overline{d_Y}(y_n,z_n)\}.$$

Now the supremum always exists, so that we get a well-defined metric. The same definition works to give a metric on the set of continuous functions $X \longrightarrow Y$. We define the sup metric on the set $\mathcal{C}(X,Y)$ of continuous functions to be

$$d_{\text{sup}}(f,g) = \sup_{x\in X} \{\overline{d_Y}(f(x),g(x))\}.$$

This is also called the **uniform metric**, for the following reason.

Proposition 3.4. *Let (f_n) be a sequence in $\mathcal{C}(X,Y)$. Then $(f_n) \to f$ in the uniform metric on $\mathcal{C}(X,Y)$ if and only if $(f_n) \to f$ uniformly.*

Given a function $f \in \mathcal{C}(X,Y)$ and a point $x \in X$, one can evaluate the function to get $f(x) \in Y$. In other words, we have an evaluation function

$$\text{eval} : \mathcal{C}(X,Y) \times X \longrightarrow Y.$$

Proposition 3.5. *Consider $\mathcal{C}(X,Y) \times X$ as a metric space using the max metric. Then* eval *is continuous.*

Proof. We know that to determine if a function between metric spaces is continuous, it it suffices to check that it takes convergent sequences to convergent sequences. Suppose that $(f_n,x_n) \to (f,x)$. We wish to show that

$$\text{eval}(f_n,x_n) = f_n(x_n) \to \text{eval}(f,x) = f(x).$$

Since $(f_n,x_n) \to (f,x)$, it follows that $f_n \to f$ and $x_n \to x$ (since the projections are continuous).

4. Fri, Sept. 5

Let $\varepsilon > 0$. Then there exists N_1 such that if $n > N_1$ then $d_{\text{sup}}(f_n,f) < \varepsilon/2$. By the definition of the sup metric, this implies that $d_Y(f_n(x_n),f(x_n)) < \varepsilon/2$. But now f is continuous, so there exists N_2 such that if $n > N_2$ then $d_Y(f(x_n),f(x)) < \varepsilon/2$. Putting these together and using the triangle inequality, if $n > N_3 = \max\{N_1,N_2\}$ then $d_Y(f_n(x_n),f(x)) < \varepsilon$. ■

Proposition 4.1. *Suppose $\varphi : X \times Y \longrightarrow Z$ is continuous. For each $x \in X$, define $\hat{\varphi}(x) : Y \longrightarrow Z$ by $\hat{\varphi}(x)(y) = \varphi(x,y)$. The function $\hat{\varphi}(x)$ is continuous.*

Proof. This could certainly be done directly, using convergence of sequences to test for continuity. Here is another way to do it, using the universal property of products.

Note that $\hat{\varphi}(x)$ can be written as the composition $Y \xrightarrow{i_x} X \times Y \xrightarrow{\varphi} Z$. By assumption, φ is continuous, so it suffices to know that $i_x : Y \to X \times Y$ is continuous. But recall that continuous maps into a product correspond precisely to a pair of continuous maps into each factor. The pair of maps here is the constant map $Y \longrightarrow X$ at x and the identity map $Y \longrightarrow Y$. The identity map is clearly continuous, and the constant map is continuous since if $U \subseteq X$ is open, then the preimage under the constant map is either (1) all of Y if $x \in U$ or (2) empty if $x \notin U$. So it follows that i_x is continuous. ∎

We are headed to the universal property of the mapping space. Keeping the notation from above, given a continuous function

$$\varphi : X \times Y \longrightarrow Z,$$

we get a function

$$\hat{\varphi} : X \longrightarrow \mathcal{C}(Y, Z).$$

Conversely, given the function $\hat{\varphi}$, we define φ by

$$\varphi(x, y) = \hat{\varphi}(x)(y).$$

Proposition 4.2. *The function φ above is continuous if $\hat{\varphi}$ is continuous.*

Proof. On homework 2. ∎

Of course, we would like this to be an if and only if, but that is only true under additional hypotheses (like Y compact, for instance.) Another way to state the if-and-only-if version of this proposition is that we get a bijection

$$\mathcal{C}(X \times Y, Z) \cong \mathcal{C}(X, \mathcal{C}(Y, Z)).$$

For those who have seen the $(\otimes, \mathrm{Hom})$ adjunction in algebra, this is completely analogous.

Quotients Another important construction that we will discuss when we move on to topological spaces is that of a quotient, or identification space. A standard example is the identification, on the unit interval $[0, 1]$, of the two endpoints. Glueing these together gives a circle S^1, and the surjective continuous map

$$e^{2\pi i x} : [0, 1] \longrightarrow S^1$$

is called the quotient map. Here the desired universal property is that if $f : [0, 1] \longrightarrow Y$ is a continuous map to another metric space such that $f(0) = f(1)$, then the map f should "factor" through the quotient. Quotients become quite complicated to express in the world of metric spaces.

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Now that we have spent some time with metric spaces, let's turn to the more general world of topological spaces.

Definition 4.3. A **topological space** is a set X with a collection of subsets $\mathcal{T}$ of X such that

(1) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$
(2) If $U_1, U_2 \in \mathcal{T}$, then $U_1 \cap U_2 \in \mathcal{T}$
(3) If $U_i \in \mathcal{T}$ for all i in some index set I, then $\bigcup_{i \in I} U_i \in \mathcal{T}$.

The collection $\mathcal{T}$ is called the **topology** on X, and the elements of $\mathcal{T}$ are referred to as the "open sets" in the topology.

Example 4.4. (1) (Metric topology) Any metric space is a topological space, where $\mathcal{T}$ is the collection of metric open sets

(2) (Discrete topology) In the discrete topology, *every* subset is open. We already saw the discrete metric on any set, so in fact this is an example of a metric topology as well.

(3) (Trivial topology) In the trivial topology, $\mathcal{T} = \{\emptyset, X\}$. That is, $\emptyset$ and X are the only empty sets. This topology does not come from a metric (unless X has fewer than two points).

(4) It is simple to write down various topologies on a finite set. For example, on the set

$$X = \{1, 2\},$$

there are 4 possible topologies. In addition to the trivial and discrete topologies, there is also

$$\mathcal{T}_1 = \{\emptyset, \{1\}, X\}$$

and

$$\mathcal{T}_2 = \{\emptyset, \{2\}, X\}.$$

(5) There are many possible topologies on $X = \{1, 2, 3\}$. But not every collection of subsets will give a topology. For instance,

$$\{\emptyset, \{1, 2\}, \{1, 3\}, X\}$$

would not be a topology, since it is not closed under intersection.

At the end of class on Friday, we introduced the notion of a topology, and I asked you to think about how many possible topologies there are on a 3-element set. The answer is ... 29. The next few answers for the number of topologies on a set of size n are[1]: 355 ($n = 4$), 6942 ($n = 5$), 209527 ($n = 6$). But there is no known formula for answer in general.

When working with metric spaces, we saw that the topology was determined by the open balls. Namely, an open set was precisely a subset that could be written as a union of balls. In many topologies, there is an analogue of these basic open sets.

Definition 5.1. A **basis** for a topology on X is a collection $\mathcal{B}$ of subsets such that

(1) (Covering property) Every point of x lies in at least one basis element
(2) (Intersection property) If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then there is a third basis element B_3 such that
$$x \in B_3 \subseteq B_1 \cap B_2.$$

A basis $\mathcal{B}$ defines a topology $\mathcal{T}_{\mathcal{B}}$ by declaring the open sets to be the unions of (arbitrarily many) basis elements.

Proposition 5.2. *Given a basis $\mathcal{B}$, the collection $\mathcal{T}_{\mathcal{B}}$ is a topology.*

Proof. It is clear that open sets are closed under unions. The emptyset is a union of no basis elements, so it is open. The set X is open by the covering property: the union of all basis elements is X. Finally, we check that the intersection of two open sets is open. Let U_1 and U_2 be open. Then
$$U_1 = \bigcup_{\alpha \in A} B_\alpha, \qquad U_2 = \bigcup_{\delta \in \Delta} B_\delta.$$
We want to show that $U_1 \cap U_2$ is open. Now
$$U_1 \cap U_2 = \left(\bigcup_{\alpha \in A} B_\alpha \right) \cap \left(\bigcup_{\delta \in \Delta} B_\delta \right) = \bigcup_{\alpha \in A, \delta \in \Delta} B_\alpha \cap B_\delta.$$
It remains to show that $B_\alpha \cap B_\delta$ is open. By the intersection property of a basis, for each $x \in B_\alpha \cap B_\delta$, there is some B_x with
$$x \in B_x \subseteq B_\alpha \cap B_\delta.$$
It follows that
$$B_\alpha \cap B_\delta = \bigcup_{x \in B\alpha \cap B_\delta} B_x,$$
so we are done. ■

Example 5.3. We have already seen that metric balls form a basis for the metric topology. In the case of the discrete metric, one can take the balls with radius 1/2, which are exactly the singleton sets.

Example 5.4. For a truly new example, we take as basis on $\mathbb{R}$, the half-open intervals $[a, b)$. The resulting topology is known as the **lower limit topology** on $\mathbb{R}$.

How is this related to the usual topology on $\mathbb{R}$? Well, any open interval (a, b) can be written as a union of half-open intervals. However, the $[a, b)$ are certainly not open in the usual topology. This says that $\mathcal{T}_{\text{standard}} \subseteq \mathcal{T}_{\ell\ell}$. The lower limit topology has more open sets than the usual topology. When one topology on a set has more open sets than another, we say it is **finer**. So the lower limit

[1] These are taken from the On-Line Encyclopedia of Integer Sequences.

topology is *finer* than the usual topology on $\mathbb{R}$, and the usual topology is *coarser* than the lower limit topology.

On any set X, the discrete topology is the finest, whereas the trivial topology is the coarsest.

When a topology is generated by a basis, there is a convenient criterion for open sets.

Proposition 5.5. *(Local criterion for open sets) Let $\mathcal{T}_\mathcal{B}$ be a topology on X generated by a basis $\mathcal{B}$. Then a set $U \subseteq X$ is open if and only if, for each $x \in U$, there is a basis element $B_x \in \mathcal{B}$ with $x \in B_x \subseteq U$.*

Proof. ($\Rightarrow$) By definition of $\mathcal{T}_\mathcal{B}$, the set U is a union of basis elements, so any $x \in U$ must be contained in one of these.

($\Leftarrow$) We can write $U = \bigcup_{x \in U} B_x$. ■

This is a good time to introduce a convenient piece of terminology: given a point x of a space X, a **neighborhood** N of x in X is a subset of X containing some open set U with $x \in U \subseteq N$. Often, we will take our neighborhoods to themselves be open.

Given our discussion of continuous maps between metric spaces, it should be clear what the right notion is for maps between topological spaces.

Definition 5.6. A function $f : X \longrightarrow Y$ between topological spaces is said to be **continuous** if for every open subset $V \subseteq Y$, the preimage $f^{-1}(V)$ is open in X.

Example 5.7. Let $X = \{1, 2\}$ with topology $\mathcal{T}_X = \{\emptyset, \{1\}, X\}$ and let $Y = \{1, 2, 3\}$ with topology $\mathcal{T}_Y = \{\emptyset, \{2\}, \{3\}, \{2, 3\}, Y\}$. Which functions $X \longrightarrow Y$ are continuous?

Let's start with the open set $\{2\} \subseteq Y$. The preimage must be open, so it can either be $\emptyset$ or $\{1\}$ or X. If the preimage is X, the function is constant at 2, which is continuous.

Suppose the preimage is $\emptyset$. Then the preimage of $\{3\}$ can be either $\emptyset$ or $\{1\}$ or X. If it is $\emptyset$, we are looking at the constant function at 1, which is continuous. If $f^{-1}(3) = X$, then f is constant at 3, which is continuous. Finally, if $f^{-1}(3) = \{1\}$, then f must be the continuous function $f(1) = 3$, $f(2) = 1$.

Finally, suppose $f^{-1}(2) = \{1\}$. Then $f^{-1}(3)$ can't be $\{1\}$ or X, so the only possible continuous f has $f^{-1}(3) = \emptyset$, so that we must have $f(1) = 2$ and $f(2) = 1$.

By the way, we asserted above that constant functions are continuous. We proved this before (top of page 7) for metric spaces, but the proof given there applies verbatim to general topological spaces.

6. Wed, Sept. 10

Proposition 6.1. *Suppose $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are continuous. Then so is their composition $g \circ f : X \longrightarrow Z$.*

Proof. Let $V \subseteq Z$ be open. Then

$$\begin{aligned}(g \circ f)^{-1}(V) &= \{x \in X \mid (g \circ f)(x) \in V\} = \{x \in X \mid g(f(x)) \in V\} \\ &= \{x \in X \mid f(x) \in g^{-1}(V)\} = \{x \in X \mid x \in f^{-1}(g^{-1}(V))\} = f^{-1}(g^{-1}(V)).\end{aligned}$$

Now g is continuous, so $g^{-1}(V)$ is open in Y, and f is continuous, so $f^{-1}(g^{-1}(V))$ is open in X. ■

Another construction we can consider with continuous functions is the idea of restricting a continuous function to a subset. For instance, the natural logarithm is a nice continuous function $\ln : (0, \infty) \longrightarrow \mathbb{R}$, but we also get a nice continuous function by considering the logarithm only on $[1, \infty)$. To have this discussion here, we should think about how a subset of a space becomes a space in its own right.

Definition 6.2. Let X be a space and let $A \subseteq X$ be a subset. We define the subspace topology on A by saying that $V \subseteq A$ is open if and only if there exists some open $U \subseteq X$ with $U \cap A = V$.

Note that the open set $U \subseteq X$ is certainly not unique.

Example 6.3. (1) Let $A = \mathbb{R} \times \{0\} \subseteq \mathbb{R}^2$. Then the subspace topology on $A \cong \mathbb{R}$ is the usual topology on $\mathbb{R}$. Indeed, consider the usual basis for $\mathbb{R}^2$ consisting of open disks. Intersecting these with A gives open intervals. In general, intersecting a basis for X with a subset A gives a basis for A, and here we clearly get the usual basis for the standard topology. The same would be true if we started with max-metric basis (consisting of open rectangles).

(2) Let $A = (0, 1) \subseteq X = \mathbb{R}$. We claim that $V \subseteq A$ is open in the subset topology if and only if V is open as a subset of $\mathbb{R}$. Indeed, suppose that V is open in A. Then $V = U \cap (0, 1)$ for some open U in $\mathbb{R}$. But now both U and $(0, 1)$ are open in $\mathbb{R}$, so it follows that their intersection is as well. The converse is clear.

Note that this statement fails for the previous example. $(0, 1) \times \{0\}$ is open in A there but not open in $\mathbb{R}^2$.

(3) Let $A = (0, 1]$. Then, in the subspace topology on A, every interval $(a, 1]$, with $a < 1$ is an open set. A basis for this topology on A consists in the (a, b) with $0 \leq a < b < 1$ and the $(a, 1]$ with $0 \leq a < 1$.

(4) Let $A = (0, 1) \cup \{2\}$. Then the singleton $\{2\}$ is an open subset of A! A basis consists of the (a, b) with $0 \leq a < b \leq 1$ and the singleton $\{2\}$.

Given a subset $A \subseteq X$, there is always the inclusion function $\iota_A : A \longrightarrow X$ defined by $\iota_A(a) = a$.

Proposition 6.4. *Given a subset $A \subseteq X$ of a topological space, the inclusion ι_A is continuous. Moreover, the subspace topology on A is the coarsest topology which makes this true.*

Proof. Suppose that $U \subseteq X$ is open. Then $\iota_A^{-1}(U) = U \cap A$ is open in A by the definition of the subspace topology.

To see that this is the coarsest such topology, suppose that $\mathcal{T}'$ is a topology which makes the inclusion $\iota_A : A \longrightarrow X$. We wish to show that $\mathcal{T}'$ is finer than the subspace topology, meaning that $\mathcal{T}_A \subseteq \mathcal{T}'$, where $\mathcal{T}_A$ is the subspace topology. So let V be open in $\mathcal{T}_A$. This means there exists $U \subseteq X$ open such that $V = U \cap A = \iota_A^{-1}(U)$. Since ι_A is continuous according to $\mathcal{T}'$, it follows that V is open in $\mathcal{T}'$. ■

Getting back to our motivational question, suppose that $f : X \longrightarrow Y$ is continuous and let $A \subseteq X$ be a subset. We define the restriction of f to A, denoted $f_{|A}$, by

$$f_{|A} : A \longrightarrow Y, \qquad f_{|A}(a) = f(a).$$

Proposition 6.5. *Let $f : X \longrightarrow Y$ be continuous and suppose that $A \subseteq X$ is a subset. Then the restriction $f_{|A} : A \longrightarrow Y$ is continuous.*

Proof. This is just the composition $f_{|A} = f \circ \iota_A$. ■

So far, we only discussed the notion of open set, but there is also the complementary notion of closed set.

Definition 6.6. Let X be a space. We say a subset $W \subseteq X$ is **closed** if the complement $X \setminus W$ is open.

Note that, despite what the name may suggest, closed does *not* mean "not open". For instance, the empty set is always both open (required for any topology) and closed (because the complement, X must be open). Similarly, there are many examples of sets that are neither open nor closed (for example, the interval $[0, 1)$ in the usual topology on $\mathbb{R}$).

Proposition 6.7. *Let X be a space.*

1. *$\emptyset$ and X are both closed in X*
2. *If W_1, W_2 are closed, then $W_1 \cup W_2$ is also closed*
3. *If W_i are closed for all i in some index set I, then $\bigcap_{i\in I} W_i$ is also closed.*

Proof. We prove (2). The point is that

$$X \setminus (W_1 \cup W_2) = (X \setminus W_1) \cap (X \setminus W_2).$$

This equality is known as one of the DeMorgan Laws ■

7. Fri, Sept. 12

Last time, we defined the notion of a closed set.

Example 7.1. Consider $\mathbb{R}_{\ell\ell}$, the real line equipped with the lower-limit topology. (Example 5.4). There, a half-open interval $[a, b)$ was declared to be open. It then follows that intervals of the form $(-\infty, b)$ and $[a, \infty)$ are open. But this then implies that $[a, b)$ is *closed* since its complement is the open set $(-\infty, a) \cup [b, \infty)$.

Not only does a topology give rise to a collection of closed sets satisfying the above properties, but one can also define a topology by specifying a list of closed sets satisfying the above properties.

Similarly, we can use closed sets to determine continuity.

Proposition 7.2. *Let $f : X \longrightarrow Y$. Then f is continuous if and only if the preimage of every closed set in Y is closed in X.*

Example 7.3. The "distance from the origin function" $d : \mathbb{R}^3 \longrightarrow \mathbb{R}$ is continuous (follows from HW 2). Since $\{1\} \subseteq \mathbb{R}$ is closed, it follows that the sphere $S^2 = d^{-1}(1)$ is closed in $\mathbb{R}^3$. More generally, S^{n-1} is closed in $\mathbb{R}^n$.

Example 7.4. Let X be any metric space, let $x \in X$, and let $r > 0$. Then the ball

$$B_{\leq r}(x) = \{y \in X \mid d(x, y) \leq r\}$$

is closed in X.

Remark 7.5. Note that some authors use the notation $\overline{B_r(x)}$ for the closed ball. This is a bad choice of notation, since it suggests that the closure of the open ball is the closed ball. But this is not always true! For instance, consider a set (with more than one point) equipped with the discrete metric. Then $B_1(x) = \{x\}$ is already closed, so it is its own closure. On the other hand, $B_{\leq 1}(x) = X$.

Consider the half-open interval $[a, b)$. It is neither open nor closed, in the usual topology. Nevertheless, there is a closely associated closed set, $[a, b]$. Similarly, there is a closely associated open set, (a, b). Notice the containments

$$(a, b) \subseteq [a, b) \subseteq [a, b].$$

It turns out that this picture generalizes.

Let's start with the closed set. In the example above, $[a, b]$ is the *smallest closed set containing* $[a, b)$. Why should we expect such a smallest closed set to exist in general? Recall that if we intersect arbitrarily many closed sets, we are left with a closed set.

Definition 7.6. Let $A \subseteq X$ be a subset of a topological space. We define the **closure of A in X** to be

$$\overline{A} = \bigcap_{A\subset B \text{ closed}} B.$$

Dually, we have $(a,b) \subset [a,b)$, and (a,b) is the largest open set contained in $[a,b)$.

Definition 7.7. Let $A \subseteq X$ be a subset of a topological space. We define the **interior of A in X** to be

$$\mathrm{Int}(A) = \bigcup_{A \supset U \text{ open}} U.$$

The difference of these two constructions is called the **boundary of A in X**, defined as

$$\partial A = \overline{A} \setminus \mathrm{Int}(A).$$

Example 7.8. (1) From what we have already said, it follows that $\partial[a,b) = \{a,b\}$.

(2) Let $A = \{1/n \mid n \in \mathbb{N}\} \subseteq \mathbb{R}$. Then A is not open, since no neighborhood of any $1/n$ is contained in A. This also shows that $\mathrm{Int}(A) = \emptyset$. But neither is A closed, because no neighborhood of 0 is contained in the complement of A. This implies that $0 \in \overline{A}$, and it turns out that $\overline{A} = A \cup \{0\}$. Thus $\partial A = \overline{A} = A \cup \{0\}$.

(3) Let $\mathbb{Q} \subseteq \mathbb{R}$. Similarly to the example above, $\mathrm{Int}(\mathbb{Q}) = \emptyset$. But since $\mathbb{R} \setminus \mathbb{Q}$ does not entirely contain any open intervals, it follows that $\overline{\mathbb{Q}} = \mathbb{R}$. (A subset $A \subseteq X$ is said to be **dense** in X if $\overline{A} = X$.) Thus $\partial\mathbb{Q} = \mathbb{R} \setminus \emptyset = \mathbb{R}$.

(4) Let's turn again to $\mathbb{R}_{\ell\ell}$. We saw that $[0,1)$ was already closed. What about $(0,1]$? Since $[0,1]$ is closed in the usual topology, this must be closed in $\mathbb{R}_{\ell\ell}$ as well. (Recall that the topology on $\mathbb{R}_{\ell\ell}$ is finer than the standard one). It follows that $(0,1]$ is either already closed, or its closure is $[0,1]$. We can ask, dually, whether the complement is open. But $(-\infty,0] \cup (1,\infty)$ is not open since it does not contain any neighborhoods of 0. It follows that $\overline{(0,1]} = [0,1]$ in $\mathbb{R}_{\ell\ell}$.

There is a convenient characterization of the closure, which we were implicitly using above.

Proposition 7.9 (Neighborhood criterion)**.** *Let $A \subseteq X$. Then $x \in \overline{A}$ if and only if every neighborhood of x meets A.*

Proof. ($\Rightarrow$) Suppose $x \in \overline{A}$. Then $x \in B$ for all closed sets B containing A. Let N be a neighborhood of x. Without loss of generality, we may suppose N is open. Now $X \setminus N$ is closed but $x \notin X \setminus N$, so this set cannot contain A. This means precisely that $N \cap A \neq \emptyset$.

($\Leftarrow$) Suppose every neighborhood of x meets A. Let $A \subset B$, where B is closed in X. Now $U = X \setminus B$ is an open set not meeting A, so it cannot be a neighborhood of x. This must mean that $x \notin X \setminus B$, or in other words $x \in B$. Since B was arbitrary, it follows that x lies in every such B. ■

In our earlier discussion of metric spaces, we considered convergence of sequences and how this characterized continuity. This is one statement from the theory of metric spaces that will not carry over to the generality of topological spaces.

Definition 7.10. We say that a sequence x_n in X converges to x in X if every neighborhood of x contains a tail of (x_n).

The following result follows immediately from the previous characterization of the closure.

Proposition 7.11. *Let (a_n) be a sequence in $A \subseteq X$ and suppose that $a_n \to x \in X$. Then $x \in \overline{A}$.*

Proof. We use the neighborhood criterion. Thus let U be a neighborhood of x. Since $a_n \to x$, a tail of (a_n) lies in U. It follows that $U \cap A \neq \emptyset$, so that $x \in \overline{A}$. ■

However, we will see next time that the converse fails in general.

Last time, we saw that if (a_n) is a sequence in $A \subseteq X$ and $a_n \to x$, then $x \in \overline{A}$. But the converse is not true in a general topological space. (The fact that these are equivalent in a metric space is known as the **sequence lemma**.) To see this, consider $\mathbb{R}$ equipped with the *cocountable* topology. Recall that this means that the nonempty open subsets are the cocountable ones.

Lemma 8.1. *Suppose that $x_n \to x$ in the cocountable topology on $\mathbb{R}$. Then (x_n) is eventually constant.*

Proof. Write B for the set

$$B = \{x_n \mid x_n \neq x\}.$$

Certainly B is countable, so it is closed. By construction, $x \notin B$, so $N = X \setminus B$ is an open neighborhood of x. But $x_n \to x$, so a tail of this sequence must lie in N. Since $\{x_n\} \cap N = \{x\}$, this means that a tail of this sequence is constant, in other words, the sequence is eventually constant. ■

Now consider $A = \mathbb{R} \setminus \{0\} \subseteq \mathbb{R}$ in the cocountable topology. A is not closed since the only closed proper subsets are the countable ones. It follows that A must be dense in $\mathbb{R}$. However, no sequence in A can converge to 0 since a convergent sequence must be eventually constant.

Similarly, we cannot use convergence of sequences to test for continuity in general topological spaces. For instance, consider the identity map

$$\mathrm{id} : \mathbb{R}_{\text{cocountable}} \longrightarrow \mathbb{R}_{\text{standard}},$$

where the domain is given the cocountable topology and the codomain is given the usual topology. This is not continuous, since the interval $(0, 1)$ is open in $\mathbb{R}_{\text{standard}}$ but not in $\mathbb{R}_{\text{cocountable}}$. On the other hand, the identity function takes convergent sequences in $\mathbb{R}_{\text{cocountable}}$, which are necessarily eventually constant, to convergent sequences in $\mathbb{R}_{\text{standard}}$. This follows from the following result, which you proved on HW1.

Proposition 8.2. *Let $f : X \longrightarrow Y$ be continuous. If $x_n \to x$ in X then $f(x_n) \to f(x)$ in Y.*

Proof. Suppose $x_n \to x$. Let V be an open neighborhood of $f(x)$. Then, since f is continuous, $f^{-1}(V)$ is an open neighborhood of x. Thus some tail of (x_n) lies in $f^{-1}(V)$, which means that the corresponding tail of $(f(x_n))$ lies in U. ■

However, all hope is not lost, since the following is true.

Proposition 8.3. *Let $f : X \longrightarrow Y$. Then f is continuous if and only if*

$$f(\overline{A}) \subseteq \overline{f(A)}$$

for every subset $A \subseteq X$.

Proof. ($\Rightarrow$) Assume f is continuous. Since $\overline{f(A)}$ is the intersection of *all* closed sets containing $f(A)$, it suffices to show that if B is such a closed set, then $f(\overline{A}) \subseteq B$. Well, $f(A) \subseteq B$, so

$$A = f^{-1}(f(A)) \subseteq f^{-1}(B).$$

Now f is continuous and B is closed, so by definition of the closure, we must have

$$\overline{A} \subseteq f^{-1}(B).$$

Applying f then gives $f(\overline{A}) \subseteq f(f^{-1}(B)) \subseteq B$.

($\Leftarrow$) Suppose that the above subset inclusion holds, and let $B \subseteq Y$ be closed. Let $A = f^{-1}(B)$. We wish to show that A is closed, i.e. that $\overline{A} = A$. Since $f(f^{-1}(B)) \subseteq B$, we know that

$$f(\overline{A}) \subseteq \overline{f(A)} \subseteq \overline{B} = B.$$

Applying f^{-1} gives

$$\overline{A} = f^{-1}(f(\overline{A})) \subseteq f^{-1}(B) = A.$$

It follows that A is closed. ■

Ok, so we have learned that points in $\overline{A}$ are good enough to determine continuity of functions, but these points are not necessarily limits of sequences in A. It turns out that there is an alternative characterization of these points.

Definition 8.4. Let X be a space and $A \subseteq X$. A point $x \in X$ is said to be an **accumulation point** (or **cluster point** or **limit point**) of A if

every neighborhood of x contains a point of A other than x itself.

We sometimes write A' for the set of accumulation points of A.

Example 8.5.
(1) Let $A = (0,1) \subseteq \mathbb{R}$. Then $A' = [0,1]$.
(2) Let $A = \{0,1\} \subseteq \mathbb{R}$. Then $A' = \emptyset$.
(3) Let $A = [0,1) \cup \{2\}$. Then $A' = [0,1]$.
(4) Let $A = \{1/n\} \subseteq \mathbb{R}$. Then $A' = \{0\}$.

The following result follows immediately from our neighborhood characterization of the closure of a set.

Proposition 8.6. *A point x is an acc. point of A if and only if $x \in \overline{A \setminus \{x\}}$.*

Certainly $A \setminus \{x\} \subseteq A$, and the closure operation preserves containment, so it follows that $A' \subseteq \overline{A}$. From the previous examples, we see that this need not be an equality. We also have $A \subseteq \overline{A}$, and it follows that

$$A \cup A' \subseteq \overline{A}.$$

Proposition 8.7. *For any subset $A \subseteq X$, we have*

$$A \cup A' = \overline{A}.$$

Proof. It remains to show that every point in the closure is either in A or in A'. Let $x \in \overline{A}$, but suppose that $x \notin A$. Then, by the neighborhood criterion, we have that for every neighborhood N of x, $N \cap A \neq \emptyset$. But since $x \notin A$, it follows that $N \cap (A \setminus \{x\}) \neq \emptyset$. In other words, $x \in A'$. ■

9. Wed, Sept. 17

Note that, although the motivation came from looking at sequences, there is no direct relation between accumulation points of A and limits of sequences in A.

We already saw an example of a point in the closure which is not the limit of a sequence. On the other hand, we can ask

Question 9.1. *If (a_n) is a sequence in A and $a_n \to x$, is $x \in A'$?*

Answer. No. Take $A = \{x\}$ and $a_n = x$. But, if we require that $x \notin A$, then the answer is yes.

As the example $X = \mathbb{R}^n$ suggests, sequences and closed sets are much better behaved for metric spaces.

Proposition 9.2 (The sequence lemma)**.** *Let $A \subseteq X$ and suppose that X is a metric space. Then $x \in \overline{A}$ if and only if x is the limit of a sequence in A.*

Proof. Suppose $a_n \to x$. Then either $x \in A'$, in which case we are done. Otherwise, there must be a neighborhood N of x such that $N \cap (A \setminus \{x\}) = \emptyset$. But $N \cap \{a_n\} \neq \emptyset$, so it must be that $x = a_n$ for some n, in other words $x \in A$.

On the other hand, suppose $x \in \overline{A}$. If $x \in A$, we can just take a constant sequence, so suppose not. For each n, $B_{1/n}(x)$ is a neighborhood of x, and $x \in \overline{A}$, so $B_{1/n}(x) \cap A \neq \emptyset$. Let $a_n \in B_{1/n}(x) \cap A$. Then the sequence $a_n \to x$, and $a_n \in A$ by construction. ■

The last few lectures, we have seen that closed sets are not as easily understood in general as they are in the case of metric spaces. Although we will not want to restrict ourselves to metric spaces, it will nevertheless be helpful to have some good characterizations of the ”reasonable” spaces. We mention here a few of these properties.

Definition 9.3. A space X is said to be Hausdorff (also called T_2) if, given any two points x and y in X, there are disjoint open sets U and V with $x \in U$ and $y \in V$.

This is a somewhat mild “separation property” that is held by many spaces in practice and that also has a number of nice consequences.

The Hausdorff property forces sequences to behave well, in the following sense.

Proposition 9.4. *In a Hausdorff space, a sequence cannot converge simultaneously to more than one point.*

Proof. Suppose $x_n \to x$ and $x_n \to y$. Every neighborhood of x contains a tail of x_n, as does any neighborhood of y. It follows that no neighborhood of x is disjoint from any neighborhood of y. Since X is Hausdorff, this forces $x = y$. ■

Proposition 9.5. *Every metric space is Hausdorff.*

Proof. If $x \neq y$, let $d = d(x, y) > 0$. Then the balls of radius $d/2$ centered at x and y are the needed disjoint neighborhoods. ■

However, of the (many, many) topologies on a finite set, the only one that is Hausdorff is the discrete topology. Indeed, if points are closed, then every subset is closed, as it is a finite union of points.

Another property of metric spaces that we used recently was the existence of the balls of radius $1/n$.

Definition 9.6. A space X is **first-countable** if, for each $x \in X$, there is a countable collection $\{U_n\}$ of neighborhoods of x such that any other neighborhood contains at least one of the U_n.

This was the key property used in proving that, in a metric space, an accumulation point of $A \subseteq X$ is the limit of an A-sequence.

Example 9.7. The space $X = \mathbb{R}_{\text{cocountable}}$ is not first countable. To see this, let $x \in X$ and suppose that $\{U_n\}$ is a collection of neighborhoods of x. By definition, each U_n is open and misses only countably many real numbers. Write $C_n = \mathbb{R} \setminus U_n$. Then $C = \bigcup_n C_n$ is also countable, and it follows that $U = X \setminus C$ is a neighborhood of x. But U does not contain any U_n because if $U_n \subseteq U$, this would mean that $C_n \supseteq C$. Instead, we see that $C_n \subseteq C$, so that $U \subseteq U_n$ for all n. The above argument is not quite careful enough, since all of the above inclusions could be equalities. To fix it, simply note that the countable set $\bigcup_n C_n$ cannot be all of $\mathbb{R}$, since it is countable. Let C' be the union C, but with one extra element of $\mathbb{R}$ added in. Then C' is still countable, and each C_n is strictly contained in C.

Similarly, we have

Proposition 9.8. *Let $f : X \longrightarrow Y$ be a function, where X is first-countable. Then f is continuous if and only if f takes convergent sequences in X to convergent sequences in Y.*

We will return to first-countable (and second-countable) spaces later in the course.

10. Fri, Sept. 19

Last time, there was a discussion of Hausdorff spaces. Here is one more nice consequence of this property.

Proposition 10.1. *If X is Hausdorff, then points are closed in X. (A space is called T_1 if points are closed.)*

Proof. The neighborhood crtierion for the complement $X \setminus \{x\}$ is easy to verify. ■

In Calculus, you saw functions defined piecewise, and one-sided limits were typically employed to establish continuity. There is an analogue of this type of construction for spaces.

Lemma 10.2 (Glueing/Pasting Lemma)**.** *Let $X = A \cup B$, where either (1) both A and B are open in X or (2) both A and B are closed in X. Then a function $f : X \longrightarrow Y$ is continuous if and only if the restrictions $f_{|A}$ and $f_{|B}$ are both continuous.*

Proof. ($\Rightarrow$) We already proved this in Proposition 6.5.

($\Leftarrow$) We give the proof assuming they are both open. Let $V \subseteq Y$ be open. We wish to show that $f^{-1}(V) \subseteq X$ is open. Let's restrict to A. We have $f^{-1}(V) \cap A = f_{|A}^{-1}(V)$. Since $f_{|A}$ is continuous, it follows that $f_{|A}^{-1}(V)$ is open (in A). Since A is open in X, it follows that $f_{|A}^{-1}(V)$ is also open in X. The same argument shows that $f^{-1}(V) \cap B$ is open in X. It follows that their unoin, which is $f^{-1}(V)$, is open in X. ■

Example 10.3. For example, we can use this to paste together the continuous absolute value function $f(x) = |x|$, as a function $\mathbb{R} \longrightarrow \mathbb{R}$. We get this by pasting the continuous functions $\iota : [0, \infty) \longrightarrow \mathbb{R}$, $x \mapsto x$, and $(-\infty, 0] \cong [0, \infty) \longrightarrow \mathbb{R}$, $x \mapsto -x$.

Example 10.4. Let's look at an example of a discontinuous function, for example

$$f(x) = \begin{cases} 1 & x \neq 1 \\ 2 & x = 1. \end{cases}$$

We can get this by pasting together two constant functions, but the domains are $\mathbb{R} \setminus \{1\}$ and $\{1\}$, one of which is open but not closed, and the other of which is closed but not open.

Example 10.5. Let $X = [0, 1] \cup [2, 3]$, given the subspace topology from $\mathbb{R}$. Note that in this case each of the subsets $A = [0, 1]$ and $B = [2, 3]$ is **both** open and closed, so we can specify a continuous function on X by giving a pair of continuous functions, one on A and the other on B.

Finally, we start to look at the idea of sameness. Two sets are thought of as the same if there is a bijection between them. A bijection is simply an invertible function. More generally, we have the following idea.

Definition 10.6. A "morphism" $f : X \longrightarrow Y$ is said to be an **isomorphism** if there is a $g : Y \longrightarrow X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.

Again, an isomorphism between sets is simply a bijection. In topology, this is called a **homeomorphism**. In other words, a homeomorphism is a continuous function with a continuous inverse. Since such a map is invertible, clearly it must be one-to-one and onto, but it is **not** true that every continuous bijection is a homeomorphism. Before we look at some examples, let's look at some non-examples.

Example 10.7. (1) Any time a set is equipped with two topologies, one of which is a refinement of the other, the identity map is a continuous bijection (in one direction) that is not a homeomorphism. For instance, we have the following such examples

$$\mathrm{id} : \mathbb{R} \longrightarrow \mathbb{R}_{\text{cofinite}}, \qquad \mathrm{id} : \mathbb{R}_{\text{cocountable}} \longrightarrow \mathbb{R}_{\text{cofinite}} \qquad \mathrm{id} : \mathbb{R}_{\text{discrete}} \longrightarrow \mathbb{R}$$

(2) Consider the exponential map $\exp : [0,1) \longrightarrow S^1$ given by $\exp(x) = e^{2\pi i x}$. This is a continuous bijection, but it is not a homeomorphism. Since homeomorphisms have continuous inverses, they must take open sets to open sets and closed sets to closed sets. But we see that exp does not take the open set $U = [0, 1/2)$ to an open set in S^1. The point $\exp(0) = (1, 0)$ has no neighborhood that is contained in $\exp(U)$.

11. Monday, Sept. 22

Last time, we were talking about homeomorphisms.

Example 11.1. (1) Consider $\tan : (0, \frac{\pi}{2}) \longrightarrow (0, \infty)$. This is a continuous bijection with continuous inverse (given by arctangent)

(2) Consider $\ln : (0, \infty) \longrightarrow \mathbb{R}$. This is a continuous bijection with inverse e^x. Composing homeomorphisms produces homeomorphisms, and we therefore get a homeomorphism

$$(0,1) \xrightarrow{\cong} (0, \frac{\pi}{2}) \xrightarrow{\cong} (0, \infty) \xrightarrow{\cong} \mathbb{R}.$$

(3) We similarly get a homeomorphism $\tan : [0, \frac{\pi}{2}) \xrightarrow{\cong} [0, \infty)$. It follows that we have

$$[0,1) \cong [0,\infty) \qquad \text{and} \qquad (0,1] \cong [0,\infty).$$

(4) One can similarly get $B_r^n(x) \cong \mathbb{R}^n$ for any n, r, and x.

The above example shows that there really are only three intervals, up to homeomorphism: the open interval, the half-open interval, and the closed interval.

We say that two spaces are **homeomorphic** if there is a homeomorphism between them (and write $X \cong Y$ as above). This is the notion of "sameness" for spaces. One of the major overarching questions for this course will be: how can we tell when two spaces are the same or are actually different?

A standard way to show that two spaces are not homeomorphic is to find a property that one has and the other does not. For instance every metric space is Hausdorff, so no non-Hausdorff space is the "same" as a metric space. But what property distinguishes the 3 interval types above? As we learn about more and more properties of spaces, this question will become easier to answer.

In the exponential example from last time, we noted that homeomorphisms must take open sets to open sets. Such a map is called an **open map**. Similarly, a **closed map** takes closed sets to closed sets.

Proposition 11.2. *Let $f : X \longrightarrow Y$ be a continuous bijection. The following are equivalent:*

1. *f is a homeomorphism*
2. *f is an open map*
3. *f is a closed map*

If we drop the assumption that f is bijective, it is no longer true that being an open map is equivalent to being a closed map. For example, the inclusion $(0,1) \longrightarrow \mathbb{R}$ is open but not closed, and the inclusion $[0,1] \longrightarrow \mathbb{R}$ is closed but not open.

Put on your hard hats! We turn now to the construction phase. We considered the product of metric spaces: let's define the product for spaces. We already know what property it should satisfy: we want it to be true that mapping continuously from some space Z into the product $X \times Y$ should be the same as mapping separately to X and to Y. Another way to describe this is that we want $X \times Y$ to be the "universal" example of a space with a pairs of maps to X and Y.

Well, if the projection $p_X : X \times Y \longrightarrow X$ is to be continuous, we need $p_X^{-1}(U) = U \times Y$ to be open whenever $U \subseteq X$ is open. Similarly, we need $X \times V$ to be open if $V \subseteq Y$ is open. We are forced to include these open sets, but we don't want to throw in anything extra that we don't need. In other words, we want the product topology on $X \times Y$ to be the coarsest topology containing the sets $U \times Y$ and $X \times V$.

Note that if we consider the collecion

$$\mathcal{B} = \{U \times Y\} \cup \{X \times V\},$$

this cannot be a basis because it fails the intersection property. A typical intersection is

$$(U \times Y) \cap (X \times V) = U \times V,$$

and if we consider all sets of this form, we do get a basis.

Definition 11.3. Given spaces X and Y, the **product topology** on $X \times Y$ has basis given by sets of the form $U \times V$, where U and V are open in X and Y, respectively.

This satisfies the universal property of a product. We have engineered the definition to make this so, but we will check this anyway. First, we make a little detour.

We pointed out above that if we considered the collection

$$\mathcal{B} = \{U \times Y\} \cup \{X \times V\},$$

we would not have a basis, as the intersection property failed. We remedied this by considering instead intersections of elements of $\mathcal{B}$. This is a useful idea that shows up often.

Given a set X, a collection $\mathcal{C}$ of subsets of X is called a **prebasis** for a topology on X if the collection covers X. Actually, in all of the textbooks, this is called a subbasis, but that is a terrible name, since it suggests that it is a basis. I will try to stick with the better name of prebasis.

We can then get a basis from the prebasis by considering finite intersections of prebasis elements.

Example 11.4. The collection of rays (a, ∞) and $(-\infty, b)$ give a prebasis for the standard topology on $\mathbb{R}$.

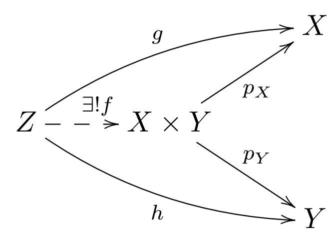
We introduced the product topology above and mentioned the universal property, but let's spend a little bit of time with it to really nail down the concept.

Theorem-Definition 11.5. *Let X and Y be spaces. Then $X \times Y$, together with the projection maps*

$$p_X : X \times Y \longrightarrow X \quad \text{and} \quad p_Y : X \times Y \longrightarrow Y,$$

satisfies the following "universal property": given any space Z and maps $g : Z \longrightarrow X$ and $h : Z \longrightarrow Y$, there is a unique continuous map $f : Z \longrightarrow X \times Y$ such that

$$g = p_X \circ f, \qquad h = p_Y \circ f.$$



Proof. The uniqueness is clear: if there exists such a continuous map f, then the conditions force this to be $f = (g, h)$. The only question is whether or not f is continuous. Consider a typical basis element $U \times V$ for the product topology on $X \times Y$. Then

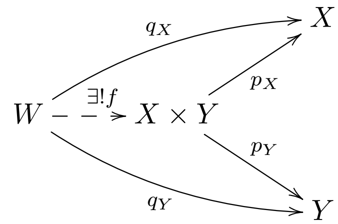
$$\begin{aligned} f^{-1}(U \times V) = \{z \in Z \mid f(z) \in U \times V\} &= \{z \in Z \mid g(z) \in U \text{ and } h(z) \in V\} \\ &= g^{-1}(U) \cap h^{-1}(V), \end{aligned}$$

which is an intersection of open sets and therefore open. ■

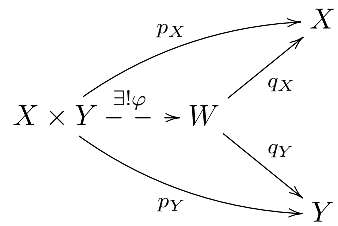
Ok, so we showed that $X \times Y$ satisfies this property, but why do we call this a "universal property"?

Proposition 11.6. *Suppose W is a space with continuous maps $q_X : W \longrightarrow X$ and $q_Y : W \longrightarrow Y$ also satisfying the property of the product. Then W is homeomorphic to $X \times Y$.*

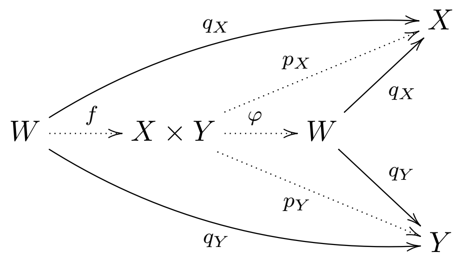
Proof. The universal property for $X \times Y$ gives us a map $f : W \longrightarrow X \times Y$.



But W also has a universal property, so we get a map $\varphi : X \times Y \longrightarrow W$ as well.



Now make Pacman eat Pacman!



We have a big diagram, but if we ignore all dotted lines, there is an obvious horizontal map $W \longrightarrow W$ to fill in the diagram, namely the id_W. Since the universal property guarantees that there is a **unique** way to fill it in, we find that $\varphi \circ f = \mathrm{id}_W$. Reversing the pacmen gives the other equality $f \circ \varphi = \mathrm{id}_{X\times Y}$. In other words, f is a homeomorphism, and $\varphi = f^{-1}$. ■

This argument may seem strange the first time you see it, but it is a typical argument that applies any time you define an object via a universal property. The argument shows that any two objects satisfying the universal property must be "the same".

Ok, so we understand $X \times Y$ as a topological space. What about a product of more than two spaces? Well, if we have a finite collection $X_1, \ldots, X_n$ of spaces, the product topology on $X_1 \times \cdots \times X_n$ has basis given by the $U_1 \times \cdots \times U_n$, or equivalently, prebasis given by the $p_j^{-1}(U_j)$. Note that this is equivalent because the basis element $U_1 \times \cdots \times U_n$, is a finite intersection of the prebasis elements $p_j^{-1}(U_j)$.

But what about the product of an *arbitrary* number of spaces? Here, the property we want is that whenever we have a space Z and maps $f_j : Z \longrightarrow X_j$ for all i, then there should be a unique continuous map $f : Z \longrightarrow \prod_{j\in J} X_j$ such that $p_j \circ f = f_j$.

Just as for finite products, we want the projection maps $p_j : \prod_{j\in J} \longrightarrow X_j$ to be continuous. This forces each $p_j^{-1}(U_j)$ to be continuous, and we can again choose these for a prebasis. We thus get a basis consisting of finite intersections $p_{j_1}^{-1}(U_{j1}) \cap \cdots \cap p_{j_k}^{-1}(U_{j_k})$.

Definition 11.7. Given spaces X_j, one for each $j \in J$, the **product topology** on $\prod_{j \in J} X_j$ has basis consisting of the $p_{j_1}^{-1}(U_{j1}) \cap \cdots \cap p_{j_k}^{-1}(U_{j_k})$.

12. Wed, Sept. 27

On the homework that was just returned, a few people used the fact that a product of continuous maps is continuous. This is true, but we have not discussed it yet, so let's do that now.

Proposition 12.1. *Let $f : X \longrightarrow Y$ and $f' : X' \longrightarrow Y'$ be continuous. Then the product map $f \times f' : X \times X' \longrightarrow Y \times Y'$ is also continuous.*

Proof. This follows very easily from the universal property. If we want to map continuously to $Y \times Y'$, it suffices to specify continuous maps to Y and Y'. The continuous map $X \times X' \longrightarrow Y$ is the composition

$$X \times X' \xrightarrow{p_X} X \xrightarrow{f} Y,$$

and the other needed map is the composition

$$X \times X' \xrightarrow{p_{X'}} X' \xrightarrow{f'} Y'.$$

■

Last time, we introduced the *product topology* on $\prod_{\alpha \in A} X_j$, which had basis

$$\mathcal{B}_{\text{prod}} = \left\{ \prod_{\alpha} U_\alpha \mid U_\alpha \subseteq X_\alpha \text{ is open, and only finitely many } U_\alpha \text{ are } \textbf{proper} \text{ subsets} \right\}.$$

Proposition 12.2. *The product topology on $\prod_{\alpha \in A} X_\alpha$, as defined above, satisfies the following universal property: given any space Z and continuous maps $f_\alpha : Z \longrightarrow X_\alpha$ for all $\alpha \in A$, there is a unique continuous $f : Z \longrightarrow \prod_{\alpha \in A} X_\alpha$ such that $p_\alpha \circ f = f_\alpha$ for all $\alpha \in A$.*

Proof. The same proof as that given in 11.5 works here. Given the maps f_α, we define f by $f(z)_\alpha = f_\alpha(z)$. Again, the equations $p_\alpha \circ f = f_\alpha$ force this choice on us. The only question is whether this makes f into a continuous map. Since the topology on $\prod_{\alpha \in A} X_\alpha$ is defined by the prebasis elements $p_\alpha^{-1}(U_\alpha)$, it suffices to show that each of these pulls back to an open set. But

$$f^{-1}(p_\alpha^{-1}(U_\alpha)) = (p_\alpha \circ f)^{-1}(U_\alpha) = f_\alpha^{-1}(U_\alpha),$$

which is open since f_α is continuous. ■

But there is another obvious guess, coming from the answer for finite products. We can think about the basis consisting of products $\prod_{\alpha} U_\alpha$. This is no longer equivalent to the product topology!

Definition 12.3. Suppose given a collection of spaces X_α. The **box topology** on $\prod_{\alpha \in A} X_\alpha$ is generated by the basis $\left\{ \prod_{\alpha \in A} U_\alpha \right\}$.

As discussed above, the box topology has more open sets; in other words, the box topology is finer than the product topology. To see that the box topology does not have the universal property we want, consider the following example: let $\Delta : \mathbb{R} \longrightarrow \prod_{n\in\mathbb{N}} \mathbb{R}$ be the diagonal map, all of whose component maps are simply the identity. For each n, let $I_n = (\frac{-1}{n}, \frac{1}{n})$. In the box topology, the subset $I = \prod_n I_n \subseteq \prod_n \mathbb{R}$ is an open set, but

$$\Delta^{-1}(I) = \bigcap_n \mathrm{id}^{-1}(I_n) = \bigcap_n I_n = \{0\}$$

is not open. So the diagonal map is not continuous in the box topology!

Since we are now considering arbitrary products, it may be useful to stop and clarify what we mean. For instance, we might want to consider a countable infinite product $\mathbb{R} \times \dots$.

Let X_α, for $\alpha \in A$, be sets. The cartesian product $\prod_{\alpha\in A} X_\alpha$ is the collection of tuples (x_α), where $x_\alpha \in X_\alpha$. This means that for each $\alpha \in A$, we want an element $x_\alpha \in X_\alpha$. In other words, we should have a function

$$x_- : A \longrightarrow X = \bigcup_\alpha X_\alpha$$

with the condition that this function satisfies $x_\alpha \in X_\alpha$. With this language, the "projection" $\prod_{\alpha\in A} X_\alpha \longrightarrow X_\alpha$ is simply the restriction along $\{\alpha\} \hookrightarrow \alpha$.

In the case that all X_α are the same set X, then $\prod_{\alpha\in A} X_\alpha$ is simply the set of functions $A \longrightarrow X$. So, the countably infinite product of $\mathbb{R}$ with itself is synonymous with the collection of sequences in $\mathbb{R}$.

Example 12.4. We mentioned above that the set of sequences in $\mathbb{R}$ is the infinite product $\prod_n \mathbb{R}$. What does a neighborhood of a sequence (x_n) look like in the product topology? We are only allowed to constrain finitely many coordinates, so a neighborhood consists of all sequences that are near to (x_n) in some fixed, finitely many coordinates.

Proposition 12.5. *Let $A_j \subseteq X_j$ for all $j \in J$. Then*

$$\prod_j \overline{A_j} = \overline{\prod_j A_j}$$

in both the product and box topologies.

Proof. As usual, we have two subsets of $\prod_j X_j$ we want to show are the same, so we establish that each is a subset of the other. The following proof works in both topologies under consideration.

($\subseteq$) Let $(x_j) \in \prod \overline{A_j}$. We use the neighborhood criterion of the closure to show that $(x_j) \in \overline{\prod_j A_j}$. Thus let $U = \prod_j U_j$ be a basic open neighborhood of (x_j). Then for each j, U_j is a neighborhood of x_j. Since $x_j \in \overline{A_j}$, it follows that U_j must meet A_j in some point, say y_j. It then follows that $(y_j) \in U \cap \prod_j A_j$. By the neighborhood criterion, it follows that $(x_j) \in \overline{\prod_j A_j}$.

($\supseteq$) Now suppose that $(x_j) \in \overline{\prod_j A_j}$. For each j, let U_j be a neighborhood of x_j. Then $p_j^{-1}(U_j)$ is a neighborhood of (x_j), so it must meet $\prod_j A_j$. But this means precisely that U_j meets A_j. It follows that $x_j \in \overline{A_j}$ for all j. ■

Note that this implies that an (arbitrary) product of closed sets is closed, using either the product or box topologies. In particular, I^2 is closed in $\mathbb{R}^2$ and T^2 is closed in $\mathbb{R}^4$.

13. Fri, Sept. 26

Proposition 13.1. *Suppose X_j is Hausdorff for each $j \in J$. Then so is $\prod_j X_j$ in both product and box topologies.*

Proof. Let $(x_j) \neq (x'_j) \in \prod_j X_j$. Then $x_\ell \neq x'_\ell$ for some particular ℓ. Since X_ℓ is Hausdorff, we can find disjoint neighborhoods U and U' of x_ℓ and x'_ℓ in X_ℓ. Then $p_\ell^{-1}(U)$ and $p_\ell^{-1}(U')$ are disjoint neighborhoods of (x_j) and (x'_j) in the product topology, so $\prod_j X_j$ is Hausdorff in the product topology.

For the box topology, we can either say that the above works just as well for the box topology, or we can say that since the box topology is a refinement of the product topology and the product topology is Hausdorff, it follows that the box topology must also be Hausdorff. ■

The converse is true as well. To see this, we use the fact that a subspace of a Hausdorff space is Hausdorff. How do we view X_ℓ as a subspace of $\prod_j X_j$? We can think about an axis inclusion. Thus pick $y_j \in X_j$ for $j \neq \ell$. We define

$$a_\ell : X_\ell \longrightarrow \prod_j X_j$$

by

$$a_\ell(x)_j = \begin{cases} x & j = \ell \\ y_j & j \neq \ell. \end{cases}$$

Note that, by the universal property of the product, in order to check that a_ℓ is continuous, it suffices to check that each coordinate map is continuous. But the coordinate maps are the identity and a lot of constant maps, all of which are certainly continuous. The map a_ℓ is certainly injective (assuming all X_j are nonempty!), and it is an example of an embedding.

Definition 13.2. A map $f : X \longrightarrow Y$ is said to be an **embedding** if it is a homeomorphism onto its image $f(X)$, equipped with the subspace topology.

We already discussed injectivity and continuity of the axis inclusion a_ℓ, so it only remains to show this is open, as a map to $a_\ell(X_\ell)$. Let $U \subseteq X_\ell$ be open. Then

$$a_\ell(U) = p_\ell^{-1}(U) \cap a_\ell(X_\ell),$$

so $a_\ell(U)$ is open in the subspace topology on $a_\ell(X_\ell)$.

We will often do the above sort of exercise: if we introduce a new property or construction, we will ask how well this interacts with other constructions/properties.

Here is another example of an embedding.

Example 13.3. Let $f : X \longrightarrow Y$ be continuous and define the graph of f to be

$$\Gamma(f) = \{(x, y) \mid y = f(x)\} \subseteq X \times Y.$$

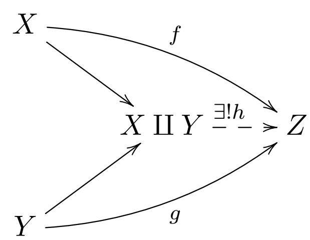
The function

$$\gamma : X \longrightarrow X \times Y, \qquad \gamma(x) = (x, f(x))$$

is an embedding with image $\Gamma(f)$.

Let us verify that this is indeed an embedding. Injectivity is easy (this follows from the fact that one of the coordinate maps is injective), and continuity comes from the universal property for the product $X \times Y$ since id_X and f are both continuous. Note that $(p_Y)_{|\Gamma(f)}$, which is continuous since it is the restriction of the continuous projection p_Y, provides an inverse to γ.

What happens if we turn all of the arrows around in the defining property of a product? We might call such a thing a "coproduct". To be precise we would want a space that is universal among spaces equipped with maps *from* X and Y. In other words, given a space Z and maps $f : X \longrightarrow Z$ and $g : Y \longrightarrow Z$, we would want a unique map from the coproduct to Z, making the following diagram commute.



The glueing lemma gave us exactly such a description, in the case that our domain space X was made up of *disjoint* open subsets A and B. In general, the answer here is given by the **disjoint union**.

Recall that, as a set, the disjoint union of sets X and Y is the subset

$$X \amalg Y \subseteq (X \cup Y) \times \{1, 2\},$$

where $X \amalg Y = (X \times \{1\}) \cup (Y \times \{2\})$. More generally, given sets X_j for $j \in J$, their disjoint union $\coprod_j X_j$ is the subset

$$\coprod_j X_j \subseteq \left(\bigcup_j X_j \right) \times J$$

given by

$$\coprod_j X_j = \bigcup_j (X_j \times \{j\}).$$

There are natural inclusions $\iota_X : X \longrightarrow X \amalg Y$ or more generally $\iota_{X_j} : X_j \hookrightarrow \coprod_j X_j$. We topologize the coproduct by giving it the finest topology such that all ι_{X_j} are continuous. In other words, a subset $U \subseteq \coprod_j X_j$ is open if and only if $\iota_j^{-1}(U) \subseteq X_j$ is open for all j.

Note that in the case of a coproduct of two spaces, the subspace topology on $X \subseteq X \amalg Y$ agrees with the original topology on X. Furthermore, both X and Y are open in $X \amalg Y$, so the universal property for the coproduct is precisely the glueing lemma.

14. Monday, Sept. 29

On Friday, we introduced the idea of a coproduct, which is dual to the product. In the case of a space X which happens to be the union of two open, disjoint, subspaces A and B, then the glueing lemma told us that X satisfies the correct property to be the coproduct $X = A \amalg B$.

For a more general coproduct $\coprod_j X_j$, we declared $U \subseteq \coprod_j X_j$ to be open if and only if $\iota_j^{-1}(U)$ is open for all j. Let's verify that this satisfies the universal property.

Thus let $f_j : X_j \longrightarrow Z$ be continuous for all $j \in J$. It is clear that, set-theoretically, the various images $\iota_j(X_j)$ inside the coproduct are disjoint and that their union is the entire coproduct. So to define a function on the coproduct, it suffices to define a function on each $\iota_j(X_j)$. But each ι_j is injective, in other words a bijection onto its image, so defining $f_{|\iota_j(X_j)}$ is equivalent to defining $f_{|\iota_j(X_j)} \circ \iota_j$. But the latter, according to the universal property, is supposed to be f_j. So the upshot of all of this is that there is no choice in how we define the function f. As usual, we only need verify that this function f is continuous.

Let $V \subseteq Z$ be open. We wish to know that $f^{-1}(V)$ is open in $\coprod_j X_j$. But according to the topology on the coproduct, this amounts to showing that each $\iota_j^{-1} f^{-1}(V)$ is open. But this is $(f \circ \iota_j)^{-1}(V) = f_j^{-1}(V)$, which is open by the assumption that each f_j is continuous.

Example 14.1. (1) Consider $X = [0, 1]$ and $Y = [2, 3]$. Then in this case $X \amalg Y$ is homeomorphic to the subspace $X \cup Y$ of $\mathbb{R}$. The same is true of these two intervals are changed to be open or half-open.

(2) Consider $X = (0, 1)$ and $Y = \{1\}$. Then $X \amalg Y$ is **not** homeomorphic to $(0, 1) \cup \{1\} = (0, 1]$. The singleton $\{1\}$ is open in $X \amalg Y$ but not in $(0, 1]$. Instead, $X \amalg Y$ is homeomorphic to $(0, 1) \cup \{2\}$.

(3) Similarly $(0, 1) \amalg [1, 2]$ is homeomorphic to $(0, 1) \cup [2, 3]$ but not to $(0, 1) \cup [1, 2] = (0, 2]$.

(4) In yet another similar example, $(0, 2) \amalg (1, 3)$ is homeomorphic to $(0, 1) \cup (2, 3)$ but not to $(0, 2) \cup (1, 3) = (0, 3)$.

Proposition 14.2. *Let X_i be spaces, for $i \in I$. Then $\coprod_i X_i$ is Hausdorff if and only if all X_i are Hausdorff.*

Proof. This is even easier than for products. First, X_i always embeds as a subspace of the coproduct, so it follows that X_i is Hausdorff if the coproduct is as well. On the other hand, suppose all X_i are Hausdorff and suppose that $x \neq y$ are points of $\coprod_i X_i$. Either x and y come from different X_i's, in which case the X_i's themselves serve as the disjoint neighborhoods. The alternative is that x and y live in the same Hausdorff X_i, but then we can find disjoint neighborhoods in X_i. ■

The next important construction is that of a quotient, or identification space.

The general setup is that we have a surjective map $q : X \longrightarrow Y$, which we view as making an identification of points in X. More precisely, suppose that we have an equivalence relation $\sim$ on X. We can consider the set $X/\sim$ of equivalence classes in X. There is a natural surjective map $q : X \longrightarrow X/\sim$ which takes $x \in X$ to its equivalence class.

And in fact every surjective map is of this form. Suppose that $q : X \longrightarrow Y$ is surjective. We define a relation on X by saying that $x \sim x'$ if and only if $q(x) = q(x')$. Then the function $X/\sim \longrightarrow Y$ sending the class of x to $q(x)$ is a bijection.

We want to mimic the above situation in topology, but to understand what this should mean, we first look at the universal property of the quotient for sets. This says: if $f : X \longrightarrow Z$ is a function that is constant on the equivalence classes in X, then there is a (unique) factorization $g : X/\sim \longrightarrow Z$ with $g \circ q = f$.

We want to have a similar setup in topology. Said in the equivalence relation framework, given a space X and a relation $\sim$ on X, we want a continuous map $q : X \longrightarrow Y$ such that given any space Z with a continuous map $f : X \longrightarrow Z$ which is constant on equivalence classes, there is a unique continuous map $g : Y \longrightarrow Z$ such that $g \circ q = f$. By considering the cases in which Z is a set with the trivial topology, so that maps to Z are automatically continuous, we can see that on the level of sets $q : X \longrightarrow Y$ must be $X \longrightarrow X/\sim$. It remains only to specify the topology on $Y = X/\sim$.

We want the topological quotient to be the universal example of a continuous map out of X which is constant on equivalence classes. Universal here means that we always want to have a map $Y \longrightarrow Z$ whenever $f : X \longrightarrow Z$ is another such map. Since we want to construct maps *out of* Y, this suggests we should include as many open sets as possible in Y. This leads to the following definition.

Definition 14.3. We say that a surjective map $q : X \longrightarrow Y$ is a **quotient map** if $V \subseteq Y$ is open if and only if $q^{-1}(V)$ is open in X.

One implication is the definition of continuity, but the other is given by our desire to include as many opens as we can.

Proposition 14.4. *(Universal property of the quotient) Let $q : X \longrightarrow Y$ be a quotient map. If Z is any space, and $f : X \longrightarrow Z$ is any continuous map that is constant on the fibers*[2] *of q, then there exists a unique continuous $g : Y \longrightarrow Z$ such that $g \circ q = f$.*

Proof. It is clear how g must be defined: $g(y) = f(x)$ for any $x \in q^{-1}(y)$. It remains to show that g is continuous. Let $W \subseteq Z$ be open. We want $g^{-1}(W) \subseteq Y$ to be open as well. By the definition of a quotient map, $g^{-1}(W)$ is open if and only if $q^{-1}(g^{-1}(W)) = (g \circ q)^{-1}(W) = f^{-1}(W)$ is open, so we are done by continuity of f. ■

Example 14.5. Define $q : \mathbb{R} \longrightarrow \{-1, 0, 1\}$ by

$$q(x) = \begin{cases} 0 & x = 0 \\ \frac{|x|}{x} & x \neq 0. \end{cases}$$

What is the resulting topology on $\{-1, 0, 1\}$? The points -1 and 1 are open, and the only open set containing 0 is the whole space.

Note that this example shows that a quotient of a Hausdorff space need not be Hausdorff.

Proposition 14.6. *Let $q : X \longrightarrow Y$ be a continuous, surjective, open map. Then q is a quotient map. The same is true if q is closed instead of open.*

Proof. One implication is simply the definition of continuity. For the other, suppose that $V \subseteq Y$ is a subset such that $q^{-1}(V) \subseteq X$ is open. Then $q(q^{-1}(V))$ is open since q is open. Finally, we have $V = q(q^{-1}(V))$ since q is surjective. ■

The converse is not true, however, as the next example shows.

Example 14.7. Consider $q; \mathbb{R} \longrightarrow [0, \infty)$ given by

$$q(x) = \begin{cases} 0 & x \leq 0 \\ x & x \geq 0. \end{cases}$$

The quotient topology on $[0, \infty)$ is the same as the subspace topology it gets from $\mathbb{R}$. But this is not an open map, since the image of $(-2, -1)$ is $\{0\}$, which is not open.

[2] A "fiber" is simply the preimage of a point.

We discussed last time the fact that a quotient map need not be open. Nevertheless, there is a class of open sets that are always carried to open sets.

Definition 15.1. Let $q : X \longrightarrow Y$ be a continuous surjection. We say a subset $A \subseteq X$ is **saturated** (with respect to q) if it is of the form $q^{-1}(V)$ for some subset $V \subseteq Y$.

It follows that A is saturated if and only if $q^{-1}(q(A)) = A$. Recall that a **fiber** of a map $q : X \longrightarrow Y$ is the preimage of a single point. Then another description is that A is saturated if and only if it contains all fibers that it meets.

Proposition 15.2. *A continuous surjection $q : X \twoheadrightarrow Y$ is a quotient map if and only if it takes saturated open sets to saturated open sets.*

Proof. Exercise. ■

Last time, we defined the quotient topology coming from a continuous surjection $q : X \longrightarrow Y$. Recall that q is a quotient map (and Y has the quotient topology) if $V \subseteq Y$ is open precisely when $q^{-1}(V) \subseteq X$ is open.

Example 15.3. (Collapsing a subspace) Let $A \subseteq X$ be a subspace. We define a relation on X as follows: $x \sim y$ if both are points in A or if neither is in A and $x = y$. Here, we have one equivalence class for the subset A, and every point outside of A is its own equivalence class. Standard notation for the set $X/\sim$ of equivalence classes under this relation is X/A. The universal property can be summed up as saying that any map on X which is constant on A factors through the quotient X/A.

For example, we considered last time the example $\mathbb{R}/(-\infty, 0] \cong [0, \infty)$.

Example 15.4. Consider $\partial I \subseteq I$. The exponential map $e : I \longrightarrow S^1$ is constant on ∂I, so we get an induced continuous map $\varphi : I/\partial I \longrightarrow S^1$, which is easily seen to be a bijection. In fact, it is a homeomorphism. Once we learn about compactness, it will be easy to see that this is a closed map.

We show instead that it is open. A basis for $I/\partial I$ is given by $q(a, b)$ with $0 < a < b < 1$ and by $q([0, a) \cup (b, 1])$ with again $0 < a < b < 1$. It is clear that both are taken to basis elements for the subspace topology on S^1. It follows that φ is a homeomorphism.

Example 15.5. Generalizing the previous example, for any closed ball $D^n \subseteq \mathbb{R}^{n+1}$, we can consider the quotient $D^n/\partial D^n$. Exercise: define a surjective continuous map

$$q : D^n \longrightarrow S^n$$

taking the origin to the south pole and the boundary to the north pole. This then defines a continuous bijection $D^n/\partial D^n \longrightarrow S^n$, and we will see later in the course that this is automatically a homeomorphism.

Example 15.6. (Real projective space) On S^n we impose the equivalence relation $\mathbf{x} \sim -\mathbf{x}$. The resulting quotient space is known as n-dimensional real projective space and is denoted $\mathbb{RP}^n$.

Consider the case $n = 1$. We have the hemisphere inclusion $I \hookrightarrow S^1$ given by $x \mapsto e^{ix\pi}$. Then the composition $I \hookrightarrow S^1 \twoheadrightarrow \mathbb{RP}^1$ is a quotient map that simply identifies the boundary ∂I to a point. In other words, this is example 15.3 from above, and we conclude that $\mathbb{RP}^1 \cong S^1$. However, the higher-dimensional versions of these spaces are certainly not homeomorphic.

Example 15.7. (Complex projective space) Consider S^{2n-1} as a subspace of $\mathbb{C}^n$. We then have the coordinate-wise multiplication by elements of $S^1 \cong U(1)$ on $\mathbb{C}^n$. This multiplication restricts to a multiplication on the subspace S^{2n-1}, and we impose an equivalence relation $(z_1, \dots, z_n) \sim (\lambda z_0, \dots, \lambda z_n)$ for all $\lambda \in S^1$. The resulting quotient space is the complex projective space $\mathbb{CP}^n$.

Example 15.8. (Torus) On $I \times I$, we impose the relation $(0, y) \sim (1, y)$ and also the relation $(x, 0) \sim (x, 1)$. The resulting quotient space is the torus $T^2 = S^1 \times S^1$. We recognize this as the product of two copies of example 15.3, but beware that in general a product of quotient maps need not be a quotient map.

A number of the examples above have secretly been examples of a more general construction, namely the quotient under the action of a group.

Definition 15.9. A **topological group** is a based space (G, e) with a continuous multiplication $m : G \times G \longrightarrow G$ and inverse $i : G \longrightarrow G$ satisfying all of the usual axioms for a group.

Remark 15.10. Munkres requires all topological groups to satisfy the condition that points are closed. We will not make this restriction, though the examples we will consider will all satisfy this.

Example 15.11.

(1) Any group G can be considered as a topological group equipped with the discrete topology. For instance, we have the cyclic groups $\mathbb{Z}$ and $C_n = \mathbb{Z}/n\mathbb{Z}$.
(2) The real line $\mathbb{R}$ is a group under addition, This is a topological group because addition and multiplication by -1 are both continuous. Note that here $\mathbb{Z}$ is at the same time both a subspace and a subgroup. It is thus a topological subgroup.
(3) If we remove zero, we get the multiplicative group $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$ of real numbers.
(4) Inside $\mathbb{R}^\times$, we have the subgroup $\{1, -1\}$ of order two.
(5) $\mathbb{R}^n$ is also a topological group under addition. In the case $n = 2$, we often think of this as $\mathbb{C}$.
(6) Again removing zero, we get the multiplicative group $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ of complex numbers.
(7) Inside $\mathbb{C}^\times$ we have the subgroup of complex numbers of norm 1, aka the circle group $S^1 \cong U(1) = SO(2)$.
(8) This last example suggests that matrix groups in general are good candidates. For instance, we have the topological group $Gl_n(\mathbb{R})$. This is a subspace of $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$. The determinant mapping $\det : M_n(\mathbb{R}) \longrightarrow \mathbb{R}$ is polynomial in the coefficients and therefore continuous. The general linear group is the complement of $\det^{-1}(0)$. It follows that $Gl_n(\mathbb{R})$ is an open subspace of $\mathbb{R}^{n^2}$.
(9) Inside $Gl_n(\mathbb{R})$, we have the closed subgroups $Sl_n(\mathbb{R})$, $O(n)$, $SO(n)$.

16. Fri, Oct. 3

Let G be a topological group and fix some $h \in G$. Define $L_h : G \longrightarrow G$ by $L_h(g) = hg$. This is left multiplication by h. The definition of topological group implies that this is continuous, as L_h is the composition

$$G \xrightarrow{(h,\mathrm{id})} G \times G \xrightarrow{m} G.$$

Moreover, $L_{h^{-1}}$ is clearly inverse to L_h and continuous by the same argument, so we conclude that each L_h is a homeomorphism. Since $L_h(e) = h$, we conclude that neighborhoods around h look like neighborhoods around e. Since h was arbitrary, we conclude that neighborhoods around one point look like neighborhoods around any other point. This implies that a space like the unoin of the coordinate axes in $\mathbb{R}^2$ cannot be given the structure of topological group, as neighborhoods around the origin do not resemble neighborhoods around other points.

The main reason for studying topological groups is to consider their *actions* on spaces.

Definition 16.1. Let G be a topological group and X a space. A **left action** of G on X is a map $a : G \times X \longrightarrow X$ which is associative and unital. This means that $a(g, a(h, x)) = a(gh, x)$ and

$a(e,x) = x$. Diagrammatically, this is encoded as the following commutative diagrams

$$\begin{array}{ccc} G \times G \times X & \xrightarrow{\mathrm{id}\times a} & G \times X \\ \downarrow{\scriptstyle m\times \mathrm{id}} & & \downarrow{\scriptstyle a} \\ G \times X & \xrightarrow[a]{} & X \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{e,\mathrm{id}} & G \times X \\ & \searrow^{\mathrm{id}} & \downarrow{\scriptstyle a} \\ & & X. \end{array}$$

It is common to write $g \cdot x$ or simply gx rather than $a(g,x)$.

There is a similar notion of right action of G on X, given by a map $X \times G \longrightarrow X$ satisfying the appropriate properties.

Proposition 16.2. *Suppose that $(g,x) \mapsto g \cdot x$ is a left action of G on X. Then the assignment $(x,g) \mapsto g^{-1} \cdot x$ defines a right action of G on X.*

Proof. The only point of interest is the associativity property. We write $x \cdot g = g^{-1} \cdot x$. Then

$$(x \cdot g) \cdot h = h^{-1} \cdot (g^{-1} \cdot x) = (h^{-1}g^{-1}) \cdot x = (gh)^{-1} \cdot x = x \cdot (gh),$$

which verifies that we have a right action. ■

Given an action of G on a space X, we define a relation on X by $x \sim y$ if $y = g \cdot x$ for some g. The equivalence classes are known as **orbits** of G in X, and the quotient of X by this relation is typically written as X/G. Really, the notation X/G should be reserved for the quotient by a *right action* of G on X, and the quotient by a left action should be $G\backslash X$.

Example 16.3. (1) For any G, left multiplication gives an action of G on itself! This is a transitive action, meaning that there is only one orbit, and the quotient $G\backslash G$ is just a point.

Note that we saw above that, for each $h \in G$, the map $L_h : G \longrightarrow G$ is a homeomorphism. This generalizes to any action. For each $g \in G$, the map $a(g,-) : X \longrightarrow X$ is a homeomorphism.

(2) For any (topological) subgroup $H \leq G$, left multiplication by elements of H gives a left action of H on G. Note that an orbit here is precisely a right coset Hg. The quotient is $H\backslash G$, the set of right cosets of H in G.

(3) The following example is interesting not for topological reasons but rather for the left action/right action distinction. Let X be a space, n a natural number, and Σ_n the symmetric group on n letters. Then there is a natural action of Σ_n on X^n. In the literature, this is often described as a left action, but the simpler action that arises is a right action.

Note that Σ_n is the automorphism group (group of self-bijections) of the set $\mathbf{n} = \{1,\ldots,n\}$. We can regard X^n as the set of functions $\mathbf{n} \xrightarrow{x_{(-)}} X$. There is an obvious way to combine a bijection and a function, via composition. The assignment $(x,\sigma) \mapsto x \circ \sigma$ defines a right action of Σ_n on X^n.

As I mentioned, in the literature, there is frequent reference to a left action, but this is simply the left action $\sigma \cdot x_{(-)} := x_{(-)} \cdot \sigma^{-1}$. In other words,

$$\sigma \cdot (x_1,\ldots,x_n) = (x_{\sigma^{-1}(1)},\ldots,x_{\sigma^{-1}(n)}).$$

(4) Consider the subgroup $\mathbb{Z} \leq \mathbb{R}$. Since $\mathbb{R}$ is abelian, we don't need to worry about about left vs. right actions or left vs. right cosets. We then have the quotient $\mathbb{R}/\mathbb{Z}$, which is again a topological group (again, $\mathbb{R}$ is abelian, so $\mathbb{Z}$ is normal).

What is this group? Once again, consider the exponential map $\exp : \mathbb{R} \longrightarrow S^1$ given by $\exp(x) = e^{2\pi i x}$. This is surjective, and it is a homomorphism since

$$\exp(x+y) = \exp(x)\exp(y).$$

The First Isomorphism Theorem in group theory tells us that $S^1 \cong \mathbb{R}/\ker(\exp)$, at least as a group. The kernel is precisely $\mathbb{Z} \leq \mathbb{R}$, and it follows that $S^1 \cong \mathbb{R}/\mathbb{Z}$ as a group. To see that this is also a homeomorphism, we need to know that $\exp : \mathbb{R} \longrightarrow S^1$ is a quotient map, but this follows from our earlier verification that $I \longrightarrow S^1$ is a quotient. Another way to think about this is that the universal property of the quotient gives us continuous maps $I/\partial I \longrightarrow \mathbb{R}/\mathbb{Z} \longrightarrow I/\partial I$ which are inverse to each other.

17. Mon, Oct. 6

(5) Similarly, we can think of $\mathbb{Z}^n$ acting on $\mathbb{R}^n$, and the quotient is $\mathbb{R}^n/\mathbb{Z}^n \cong (S^1)^n = T^n$.

(6) The group $Gl(n)$ acts on $\mathbb{R}^n$ (just multiply a matrix with a vector), but this is not terribly interesting, as there are only two orbits: the origin is a closed orbit, and the complement is an open orbit. Thus the quotient space consists of a closed point and an open point.

(7) More interesting is the action of the subgroup $O(n)$ on $\mathbb{R}^n$. Using the fact that orthogonal matrices preserve norms, it is not difficult to see that the orbits are precisely the spheres around the origin. We claim that the quotient is the space $[0, \infty)$ (thought of as a subspace of $\mathbb{R}$).

To see this, consider the continuous surjection $|-| : \mathbb{R}^n \longrightarrow [0, \infty)$. By considering how this acts on open balls, you can show that this is an open map and therefore a quotient. But the fibers of this map are precisely the spheres, so it follows that this is the quotient induced by the above action of $O(n)$.

At the end of class last time, we were looking at the example of $O(n)$ acting on $\mathbb{R}^n$, and we claimed that the quotient was $[0, \infty)$. We saw that the relation coming from the $O(n)$-action was the same as that coming from the surjection $\mathbb{R}^n \longrightarrow [0, \infty)$. Namely, we identify points if and only if they have the same norm. To see that the quotient by the $O(n)$-action is homeomorphic to $[0, \infty)$, it remains to show that the norm map $\mathbb{R}^n \longrightarrow [0, \infty)$ is a quotient map. We know already that it is a continuous surjection, and by considering basis elements (open balls) in $\mathbb{R}^n$, we can see that it is open as well. We leave this verification to the industrious student!

Why does the above argument show that the quotient $\mathbb{R}^n/O(n)$ is homeomorphic to $[0, \infty)$. We now have two quotient maps out of $\mathbb{R}^n$, and they are defined using the same equivalence relation on $\mathbb{R}^n$. By the universal property of quotients, the two spaces are homeomorphic!

Let's get on with more examples.

Example 17.1. (1) Let $\mathbb{R}^\times$ act on $\mathbb{R}^n$ via scalar multiplication. This action preserves lines, and within each line there are two orbits, one of which is the origin. Note that the only saturated open set containing 0 is $\mathbb{R}^n$, so the only neighborhood of 0 in the quotient is the entire space.

(2) Switching from n to $n+1$ for convenience, we can remove that troublesome 0 and let $\mathbb{R}^\times$ act on $X_{n+1} = \mathbb{R}^{n+1} \setminus \{0\}$. Here the orbits are precisely the lines (with origin removed). The quotient is $\mathbb{RP}^n$.

To see this, recall that we defined $\mathbb{RP}^n$ as the quotient of S^n by the relation $\mathbf{x} \sim -\mathbf{x}$. This is precisely the relation that arises from the action of the subgroup $C_2 = \{1, -1\} \leq \mathbb{R}^\times$ on $S^n \subseteq \mathbb{R}^{n+1}$.

Now notice that the map $\mathbb{R}^{n+1} \setminus \{0\} \longrightarrow S^n \times \mathbb{R}_{>0}$ given by $\mathbf{x} \mapsto \left(\frac{\mathbf{x}}{\|\mathbf{x}\|}, \|\mathbf{x}\|\right)$ is a homeomorphism. Next, note that we have an isomorphism $\mathbb{R}^\times \cong C_2 \times \mathbb{R}^\times_{>0}$. Thus the quotient $(\mathbb{R}^{n+1} \setminus \{0\})/\mathbb{R}^\times$ can be viewed as the two step quotient $\Big((S^{n-1} \times \mathbb{R}_{>0})/\mathbb{R}^\times_{>0})\Big)/C_2$. But $(\mathbb{R}^{n-1} \times \mathbb{R}_{>0})/\mathbb{R}^\times_{>0} \cong S^{n-1}$, so we are done.

We can think of $\mathbb{RP}^n$ in yet another way. Consider the following diagram:

$$\begin{array}{ccccc} D^n & \longrightarrow & S^n & \longrightarrow & \mathbb{R}^{n+1} \setminus \{0\} \\ \downarrow & & \downarrow & & \downarrow \\ D^n/\sim & \longrightarrow & S^n/C_2 & \longrightarrow & \mathbb{R}^{n+1} \setminus \{0\}/\mathbb{R}^\times \end{array}$$

The map $D^n \longrightarrow S^n$ is the inclusion of a hemisphere. The relation on D^n is the relation $\mathbf{x} \sim -\mathbf{x}$, but only allowed *on the boundary* ∂D^n. All maps on the bottom are continuous bijections, and again we will see later that they are necessarily homeomorphisms.

Note that the relation we imposed on D^n does *not* come from an action of C_2 on D^n. Let us write $C_2 = \langle \sigma \rangle$. We can try defining

$$\sigma \cdot \mathbf{x} = \begin{cases} \mathbf{x} & \mathbf{x} \in \mathrm{Int}(D^n) \\ -\mathbf{x} & \mathbf{x} \in \partial(D^n), \end{cases}$$

where here the interior and boundary are taken in S^n. But this is not continuous, as the convergent sequence

$$\left(\sqrt{1 - \frac{1}{n}}, 0, \dots, 0, \sqrt{\frac{1}{n}} \right) \to (1, 0, \dots, 0)$$

is taken by σ to a convergent sequence, but the new limit is not $\sigma(1, 0, \dots, 0) = (-1, 0, \dots, 0)$.

18. Wed, Oct. 8

(3) We have a similar story for $\mathbb{CP}^n$. There is an action of $\mathbb{C}^\times$ on $\mathbb{C}^{n+1} \setminus \{0\}$, and the orbits are the punctured complex lines. We claim that the quotient is $\mathbb{CP}^n$.

We defined $\mathbb{CP}^n$ as a quotient of an S^1-action on S^{2n+1}. We also have a homeomorphism $\mathbb{C}^{n+1} \setminus \{0\} \cong S^{2n+1} \times \mathbb{R}_{>0}$ and an isomorphism $\mathbb{C}^\times \cong S^1 \times \mathbb{R}^\times_{>0}$. We can then describe $\mathbb{CP}^n$ as the two-step quotient

$$\left(\mathbb{C}^{n+1} \setminus \{0\}\right)/\mathbb{C}^\times \cong \left((S^{2n+1} \times \mathbb{R}_{>0})/\mathbb{R}^\times_{>0}\right)/S^1 \cong S^{2n+1}/S^1 = \mathbb{CP}^n.$$

We have been studying actions of topological groups on spaces, and the resulting quotient spaces X/G. But there is another way to think about this material. Suppose you have a set Y that you would like to topologize. One way to create a topology on Y is as follows. Pick a point $y_0 \in Y$. If there is a transitive action of some topological group G on Y, then the orbit-stabilizer theorem asserts that Y can be identified with G/H, where $H \leq G$ is the stabilizer subgroup consisting of all $h \in G$ such that $h \cdot y_0 = y_0$. But G/H is a topological space, so we define the topology on Y to be the one coming from the bijection $Y \cong G/H$.

Example 18.1. (Grassmannian) We saw that the projective spaces can be identified with the set of lines in $\mathbb{R}^n$ or $\mathbb{C}^n$, suitably topologized. We can similarly consider the set of k-dimensional linear subspaces in $\mathbb{R}^n$ (or $\mathbb{C}^n$). It is not clear how to topologize this set.

However, there is a natural action of $O(n)$ on the set of k-planes in $\mathbb{R}^n$. Namely, if $A \in O(n)$ is an orthogonal matrix and $V \subseteq \mathbb{R}^n$ is a k-dimensional subspace, then $A(V) \subseteq \mathbb{R}^n$ is another k-dimensional subspace. Furthermore, this action is transitive. To see this, it suffices to show that given any subspace V, there is a matrix taking the standard subspace $E_k = \mathrm{Span}\{\beta_1, \dots, \beta_k\}$ to V. Thus suppose $V = \mathrm{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is a k-dimensional subspace with given orthogonal basis. This can be completed to an orthogonal basis of $\mathbb{R}^n$. Then if A is the orthogonal matrix with columns the $\mathbf{v}_i$, A takes the standard subspace E_k to V.

The stabilizer of E_k is the subgroup of orthogonal matrices that take E_k to E_k. Such matrices are block matrices, with an orthogonal $k \times k$ matrix in the upper left and an orthogonal $(n-k) \times (n-k)$ matrix in the lower right. In other words, the stabilizer subgroup is $O(k) \times O(n-k)$. It follows that the set of k-planes in $\mathbb{R}^n$ can be identified with the quotient

$$\mathrm{Gr}_{k,n}(\mathbb{R}) = O(n)/O(k) \times O(n-k).$$

Note that, from this identification, it is clear that $\mathrm{Gr}_{k,n} \cong \mathrm{Gr}_{n-k,n}$. The map takes a k-plane in $\mathbb{R}^n$ to the orthogonal complement, which is an $n-k$-plane in $\mathbb{R}^n$. The corresponding map

$$O(n)/O(k) \times O(n-k) \longrightarrow O(n)/O(n-k) \times O(k)$$

is induced by a map $O(n) \longrightarrow O(n)$. This map on $O(n)$ is conjugation by a shuffle permutation that permutes k things past $n-k$ things.

There is an identical story for the complex Grasmannians, where $O(n)$ is replaced by $U(n)$.

Example 18.2. (Flag varieties) Continuing (why not?) in this vein, we can consider the sets of flags in $\mathbb{R}^n$ or $\mathbb{C}^n$. Recall that a flag is a chain of strict inclusions of linear subspaces $0 \leq V_1 \leq V_2 \leq \cdots \leq V_k = \mathbb{R}^n$. A flag is said to be **complete** if $\dim V_k = k$. The general linear group $\mathrm{Gl}_n(\mathbb{R})$ acts transitively on the set of complete flags. Indeed, there is the standard complete flag $0 \leq E_1 \leq E_2 \leq \dots$, where $E_k = \mathrm{Span}\{\beta_1, \dots, \beta_k\}$, as above. Let $0 \leq V_1 \leq V_2 \leq \dots$ be any other complete flag. Then if we choose a basis $\{\mathbf{v}_i\}$ for $\mathbb{R}^n$ such that $V_k = \mathrm{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$, then it follows that the matrix A having the $\mathbf{v}_i$ for columns will take E_k to V_k.

In order to obtain a description of the complete flag variety $F(\mathbb{R}^n)$ as a space, we need to identify the stabilizer subgroup of a point. Let's look at the stabilizer of the standard complete flag. We want to know which matrices A will satisfy $A(\beta_k) \in E_k$ for all k. The vector $A(\beta_1)$ must be a nonzero multiple of β_1, so the only nonzero entry in the first column of A is the top left entry. A similar analysis of the other columns shows that A must be upper-triangular (and nonsingular). If we thus denote the subgroup of upper-triangular matrices by B_n (B is for 'Borel'), then we see that the flag variety can be identified with the topological space

$$F(\mathbb{R}^n) \cong \mathrm{Gl_n}(\mathbb{R})/B_n.$$

For some purposes, it is more convenient to work with the orthogonal group rather than the general linear group. This presents no real difficulty. We work with an orthonormal basis rather than any basis. Here we can see that the stabilizer of the standard flag consists of upper triangular orthogonal matrices, which coincides with the group of diagonal orthogonal matrices. These can only have 1 or -1 on the diagonal. We conclude that

$$F(\mathbb{R}^n) \cong O(n)/C_2 \times C_2 \times \cdots \times C_2.$$

There is a complex analogue as well. We have

$$F(\mathbb{C}^n) \cong \mathrm{Gl_n}(\mathbb{C})/B_n(\mathbb{C}) \cong U(n)/S^1 \times \cdots \times S^1 = U(n)/T^n.$$

What about non-complete flags? It is clear that if $\{V_i\}$ is a flag, then $\{AV_i\}$ will have the same "signature" (sequence of dimensions). But similar arguments to those above show that the general linear group or orthogonal group act transitively on the set of flags of a given signature, and we have

$$F(d_1, \dots, d_k; \mathbb{R}^n) \cong \mathrm{Gl_n}(\mathbb{R})/B_{n_1,\dots,n_k} \cong O(n)/O(n_1) \times O(n_2) \times \cdots \times O(n_k),$$

where $n_i = d_i - d_{i-1}$ and $B_{n_1,\dots,n_k}$ is the set of block-upper triangular matrices (with blocks of size n_1, n_2, etc.). Similarly,

$$F(d_1, \dots, d_k; \mathbb{C}^n) \cong \mathrm{Gl_n}(\mathbb{C})/B_{n_1,\dots,n_k}(\mathbb{C}) \cong U(n)/U(n_1) \times \cdots \times U(n_k).$$

19. Fri, Oct. 10

Exam day!

20. Mon, Oct. 13

What we have done so far corresponds roughly to Chapters 2 & 3 of Lee. Now we turn to Chapter 4.

The first idea is connectedness. Essentially, we want to say that a space cannot be decomposed into two disjoint pieces.

Definition 20.1. A **disconnection** (or separation) of a space X is a pair of disjoint, nonempty open subsets $U, V \subseteq X$ with $X = U \cup V$. We say that X is **connected** if it has no disconnection.

Example 20.2.
1. If X is a discrete space (with at least two points), then any pair of disjoint nonempty subsets gives a disconnection of X.
2. Let X be the subspace $(0,1) \cup (2,3)$ of $\mathbb{R}$. Then X is disconnected.
3. More generally, if $X \cong A \coprod B$ for nonempty spaces A and B, then X is disconnected.
4. Another example of a disconnected subspace of $\mathbb{R}$ is the subspace $\mathbb{Q}$. A disconnection of $\mathbb{Q}$ is given by $(-\infty, \pi) \cap \mathbb{Q}$ and $(\pi, \infty) \cap \mathbb{Q}$.
5. Any set with the trivial topology is connected, since there is only one nonempty open set.
6. Of the 29 topologies on $X = \{1, 2, 3\}$, 19 are connected, and the other 10 are disconnected. For example, the topology $\{\emptyset, \{1\}, X\}$ is connected, but $\{\emptyset, \{1\}, \{2, 3\}, X\}$ is not.
7. If X is a space with the generic point (or included point) topology, in which the nonempty open sets are precisely the ones containing a special point x_0, then X is connected.
8. If X is a space with the excluded point topology, in which the open proper subsets are the ones missing a special point x_0, then X is connected.
9. The lower limit topology $\mathbb{R}_{\ell\ell}$ is disconnected, as the basis elements $[a, b)$ are both open and closed (clopen!), which means that their complements are open.

Proposition 20.3. *Let X be a space. The following are equivalent:*
1. *X is disconnected*
2. *$X \cong A \coprod B$ for nonempty spaces A and B*
3. *There exists a nonempty, clopen, proper subset $U \subsetneq X$*
4. *There exists a continuous surjection $X \twoheadrightarrow \{0, 1\}$, where $\{0, 1\}$ has the discrete topology.*

Now let's look at an interesting example of a connected space.

Proposition 20.4. *The only (nonempty) connected subspaces of $\mathbb{R}$ are singletons and intervals.*

Proof. It is clear that singletons are connected. Note that, by an interval, we mean simply a convex subset of $\mathbb{R}$. It is clear that any connected subset must be an interval since if A is connected and $a < b < c$ with $a, c \in A$, then either $b \in A$ or $(-\infty, b) \cap A$ and $(b, \infty) \cap A$ give a separation of A.

So it remains to show that intervals are connected. Let $I \subseteq \mathbb{R}$ be an interval with at least two points, and let $U \subseteq I$ be nonempty and clopen. We wish to show that $U = I$. Let $a \in U$. We will show that $U \cap [a, \infty) = I \cap [a, \infty)$. A similar argument will show that $U \cap (-\infty, a] = I \cap (-\infty, a]$.

Consider the set

$$R_a = \{b \in I \mid [a, b] \subseteq U\}.$$

Note that $a \in R_a$, so that R_a is nonempty. If R_a is not bounded above, then $[a, \infty) \subseteq U \subseteq I$, and we have our conclusion. Otherwise, the set R_a has a supremum $s = \sup R_a$ in $\mathbb{R}$. Since we can express s as a limit of a U-sequence and since U is closed in I, it follows that if $s \in I$ then s must also lie in U.

Note that if $s \notin I$, then since I is an interval we have

$$U \cap [a, \infty) = [a, s) = I \cap [a, \infty).$$

On the other hand, as we just said, if $s \in I$ then $s \in U$. But U is open, so some ϵ-neighborhood of s (in I) lies in U. But no point in $(s, s + \epsilon/2)$ can lie in U (or I), since any such point would then also lie in R_a. Again, since I is an interval we have

$$U \cap [a, \infty) = [a, s] = I \cap [a, \infty).$$

■

One of the most useful results about connected spaces is the following.

Proposition 20.5. *Let $f : X \longrightarrow Y$ be continuous. If X is connected, then so is $f(X) \subseteq Y$.*

Proof. This is a one-liner. Suppose that $U \subseteq f(X)$ is closed and open. Then $f^{-1}(U)$ must be closed and open, so it must be either $\emptyset$ or X. This forces $U = \emptyset$ or $U = f(X)$. ■

Since the exponential map $\exp : [0, 1] \longrightarrow S^1$ is a continuous surjection, it follows that S^1 is connected. More generally, we have

Proposition 20.6. *Let $q : X \longrightarrow Y$ be a quotient map with X connected. Then Y is connected.*

Which of the other constructions we have seen preserve connectedness? All of them! (Well, except that subspaces of connected spaces need not be connected, as we have already seen.

Proposition 20.7. *Let $A_i \subseteq X$ be connected for each i, and assume that $x_0 \in \bigcap_i A_i \neq \emptyset$. Then $\bigcup_i A_i$ is connected.*

Proof. Assume each A_i is connected, and let $U \subseteq \bigcup_i A_i$ be nonempty and clopen. Then $x \in U$ for some $x \in \bigcup_i A_i$. Suppose $x \in A_{i_0}$. Then $U \cap A_{i_0}$ is nonempty and clopen in A_{i_0}, so $U \cap A_{i_0} = A_{i_0}$. In other words, $A_{i_0} \subseteq U$. Since $x_0 \in A_{i_0}$, it follows that $x_0 \in U$. But now for any other A_j, we have that $x_0 \in A_j \cap U$, so that $A_j \cap U$ is nonempty and clopen in A_j. It follows that $A_j \subseteq U$. ■

21. Wed, Oct. 15

Last time, we introduced the idea of connectedness and showed (1) that the connected subsets of $\mathbb{R}$ are precisely the intervals and (2) the image of a connected space under a continuous map is connected. This implies.

Theorem 21.1 (Intermediate Value Theorem)**.** *Let $f : [a, b] \longrightarrow \mathbb{R}$ be continuous. Then f attains ever intermediate value between $f(a)$ and $f(b)$.*

Proof. This follows from the fact that the image is an interval. ■

We also showed that an overlapping union of connected subspaces is connected.

As an application, we get that products interact well with connectedness.

Proposition 21.2. *Assume $X_i \neq \emptyset$ for all $i \in \{1, \dots, n\}$. Then $\prod_{i=1}^{n} X_i$ is connected if and only if each X_i is connected.*

Proof. ($\Rightarrow$) This follows from Prop 20.5, as $p_i : \prod_i X_i \longrightarrow X_i$ is surjective (this uses that all X_j are nonempty).

($\Leftarrow$) Suppose each X_i is connected. By induction, it suffices to show that $X_1 \times X_2$ is connected. Pick any $z \in X_2$. We then have the embedding $X_1 \hookrightarrow X_1 \times X_2$ given by $x \mapsto (x, z)$. Since X_1 is connected, so is its image C in the product. Now for each $x_1 \in X_1$, we have an embedding $\iota_{x_1} : X_2 \hookrightarrow X_1 \times X_2$ given by $y \mapsto (x_1, y)$. Let $D_{x_1} = \iota_{x_1}(X_2) \cup C$. Note that each D is connected, being the overlapping union of two connected subsets. But we can write $X_1 \times X_2$ as the overlapping union of all of the D_{x_1}, so by the previous result the product is connected. ■

The following result is easy but useful.

Proposition 21.3. *Let $A \subseteq B \subseteq \overline{A}$ and suppose that A is connected. Then so is B.*

Proof. Exercise ■

Theorem 21.4. *Assume $X_i \neq \emptyset$ for all $i \in I$, where is I is arbitrary. Then $\prod_i X_i$ is connected if and only if each X_i is connected.*

Proof. As in the finite product case, it is immediate that if the product is connected, then so is each factor.

We sketch the other implication. We have already showed that each finite product is connected. Now let $(z_i) \in \prod_i X_i$. For each $j \in I$, write $D_j = p_j^{-1}(z_j) \subseteq \prod_i X_i$.

For each finite collection $j_1, \ldots, j_k \in I$, let

$$F_{j_1,\ldots,j_k} = \bigcap_{j \neq j_1,\ldots,j_k} D_j \subseteq \prod_i X_i.$$

Then $F_{j_1,\ldots,j_k} \cong X_{j_1} \times \cdots \times X_{j_k}$, so it follows that $F_{j_1,\ldots,j_k}$ is connected. Now $(z_i) \in F_{j_1,\ldots,j_k}$ for every such tuple, so it follows that

$$F = \bigcup F_{j_1,\ldots,j_k}$$

is connected.

It remains to show that F is dense in $\prod_i X_i$ (in other words, the closure of F is the whole product). Let

$$U = p_{j_1}^{-1}(U_{j_1}) \cap \cdots \cap p_{j_k}^{-1}(U_{j_k})$$

be a nonempty basis element. Then U meets $f_{j_1,\ldots,j_k}$, so U meets F. Since U was arbitrary, it follows that F must be dense. ■

Note that the above proof would not have worked with the box topology. We can show directly that $\mathbb{R}^{\mathbb{N}}$, equipped with the box topology, is not connected. Consider the subset $\mathcal{B} \subset \mathbb{R}^{\mathbb{N}}$ consisting of bounded sequences. If $(z_i) \in \mathcal{B}$, then $\prod_i (z_i - 1, z_i + 1)$ is a neighborhood of (z_i) in $\mathcal{B}$. On the other hand, if $(z_i) \notin \mathcal{B}$, the same formula gives a neighborhood consisting entirely of unbounded sequences. We conclude that $\mathcal{B}$ is a nontrivial clopen set in the box topology.

Ok, so we have looked at examples and studied this notion of being connected, but if you asked your calculus students to describe what it should mean for a subset of $\mathbb{R}$ to be connected, they probably wouldn't come up with the "no nontrivial clopen subsets" idea. Instead, they would probably say something about being able to connect-the-dots. In other words, you should be able to draw a line from one point to another while staying in the subset. This leads to the following idea.

Definition 21.5. We say that $A \subseteq X$ is **path-connected** if for every pair a, b of points in A, there is a continuous function (a path) $\gamma : I \longrightarrow A$ with $\gamma(0) = a$ and $\gamma(1) = b$.

This is not unrelated to the earlier notion.

Proposition 21.6. *If $A \subseteq X$ is path-connected, then it is also connected.*

Proof. Pick a point $a_0 \in A$. For any other $b \in A$, we have a path γ_b in A from a_0 to b. Then the image $\gamma_b(I)$ is a connected subset of A containing both a_0 and b. It follows that

$$A = \bigcup_{b \in A} \gamma_b(I)$$

is connected, as it is the overlapping union of connected sets. ■

For subsets $A \subseteq \mathbb{R}$, we have

$$A \text{ is path-connected} \Rightarrow A \text{ is connected} \Leftrightarrow A \text{ is an interval} \Rightarrow A \text{ is path-connected.}$$

So the two notions coincide for subsets of $\mathbb{R}$. But the same is not true in $\mathbb{R}^2$!

Example 21.7 (Topologist's sine curve)**.** Let Γ be the graph of $\sin(1/x)$ for $x \in (0, \pi]$.Then Γ is homeomorphic to $(0, \pi]$ and is therefore path-connected and connected. Let C be the closure of Γ in $\mathbb{R}^2$. Then C is connected, as it is the closure of a connected subset. However, it is not path-connected (HW VI), as there is no path in C connecting the origin to the right end-point $(\pi, 0)$.

22. Fri, Oct. 17

Path-connectedness has much the same behavior as connectedness.

Proposition 22.1.
(1) *Images of path-connected spaces are path-connected*
(2) *Overlapping unions of path-connected spaces are path-connected*
(3) *Finite products of path-connected spaces are path-connected*

However, the topologist's sine curve shows that closures of path-connected subsets need not be path-connected.

Our proof of connectivity of $\prod_i X_i$ last time used this closure property for connected sets, so the earlier argument does not adapt easily to path-connectedness. But it turns out to be easier to prove.

Theorem 22.2. *Assume $X_i \neq \emptyset$ for all $i \in I$, where is I is arbitrary. Then $\prod_i X_i$ is path-connected if and only if each X_i is path-connected.*

Proof. The interesting direction is $(\Leftarrow)$. Thus assume that each X_i is path-connected. Let (x_i) and (y_i) be points in the product $\prod_i X_i$. Then for each $i \in \mathcal{I}$ there is a path γ_i in X_i with $\gamma_i(0) = x_i$ and $\gamma_i(1) = y_i$. By the universal property of the product, we get a continuous path

$$\gamma = (\gamma_i) : [0, 1] \longrightarrow \prod_i X_i$$

with $\gamma(0) = (x_i)$ and $\gamma(1) = (y_i)$. ■

The overlapping union property for (path-)connectedness allows us to make the following definition.

Definition 22.3. Let $x \in X$. We define the **connected component** (or simply component) of x in X to be

$$C_x = \bigcup_{\substack{x \in C \\ \text{connected}}} C.$$

Similarly, the **path-component** of X is defined to be

$$PC_x = \bigcup_{\substack{x \in P \\ \text{connected}}} P.$$

The overlapping union property guarantees that C_x is connected and that PC_x is path-connected. Since path-connected sets are connected, it follows that for any x, we have $PC_x \subseteq C_x$. An immediate consequence of the above definition(s) is that any (path-)connected subset of X is contained in some (path-)component.

Example 22.4. Consider $\mathbb{Q}$, equipped with the subspace topology from $\mathbb{R}$. Then the only connected subsets are the singletons, so $C_x = \{x\}$. Such a space is said to be **totally disconnected**.

Note that for any space X, each component C_x is closed as $\overline{C_x}$ is a connected subset containing x, which implies $\overline{C_x} \subseteq C_x$. If X has finitely many components, then each component is the complement of the finite union of the remaining components, so each component is also open, and X decomposes as a disjoint union

$$X \cong C_1 \amalg C_2 \amalg \cdots \amalg C_n$$

of its components. But this does not happen in general, as the previous example shows.

The situation is worse for path-components: they need not be open or closed, as the topologist's sine curve shows.

Definition 22.5. Let X be a space. We say that X is **locally connected** if any neighborhood U of any point x contains a connected neighborhood $x \in V \subset U$. Similarly X is **locally path-connected** if any neighborhood U of any point x contains a path-connected neighborhood $x \in V \subset U$.

This may seem like a strange definition, but it has the following nice consequence.

Proposition 22.6. *Let X be a space. The following are equivalent.*

1. *X is locally connected*
2. *X has a basis consisting of connected open sets*
3. *for every open set $U \subseteq X$, the components of U are open in X*

Proof. We show (1) $\Leftrightarrow$ (3).

Suppose that X is locally connected and let $U \subseteq X$ be open. Take $C \subseteq U$ to be a component. Let $x \in C$. We can then find a connected neighborhood $x \in V \subseteq U$. Since C is the component of x, we must have $V \subseteq C$, which shows that C is open.

Suppose, on the other hand, that (3) holds. Let U be a neighborhood of x. Then the component C_x of x in U is the desired neighborhood V. ∎

In particular, this says that the components are open if X is locally connected.

The locally path-connected property is even better.

Proposition 22.7. *Let X be a space. The following are equivalent.*

1. *X is locally path-connected*
2. *X has a basis consisting of path-connected open sets*
3. *for every open set $U \subseteq X$, the path-components of U are open in X*
4. *for every open set $U \subseteq X$, every component of U is path-connected and open in X.*

Proof. The implications (1) $\Leftrightarrow$ (3) are similar to the above. We argue for (1) $\Leftrightarrow$ (4).

Assume X is locally path-connected, and let C be a component of an open subset $U \subseteq X$. Let $P \subseteq C$ be a nonempty path-component. Then P is open in X. But all of the other path-components of C are also open, so their union, which is the complement of P, must be open. It follows that P is closed. Since C is connected, we must have $P = C$.

On the other hand, suppose that (4) holds. Let U be a neighborhood of x. Then the component C_x of x in U is the desired neighborhood V. ∎

In particular, this says that the components and path-components agree if X is locally path-connected.

Just as path-connected implies connected, locally path-connected implies locally-connected. But, unfortunately, there are no other implications between the four properties.

Example 22.8. The topologist's sine curve is connected, but not path-connected or locally connected or locally path-connected (see HWVI). Thus it is possible to be connected but not locally so.

Example 22.9. For any space X, the **cone** on X is defined to be $CX = X \times [0,1]/X \times \{1\}$. The cone on any space is always path-connected. In particular, the cone on the topologist's sine curve is connected and-path connected but not locally connected or locally path-connected.

Example 22.10. A disjoint union of two topologist's sine curves gives an example that is not connected in any of the four ways.

Example 22.11. Note that if X is locally path-connected, then connectedness is equivalent to path-connectedness. A connected example would be $\mathbb{R}$ or a one-point space. A disconnected example would be $(0,1) \cup (2,3)$ or a two point (discrete) space.

Finally, we have spaces that are locally connected but not locally path-connected.

Example 22.12. The cocountable topology on $\mathbb{R}$ is connected and locally connected but not path-connected or locally path-connected. (See HWVI)

Example 22.13. The cone on the cocountable topology will give a connected, path-connected, locally connected space that is not locally path-connected.

Example 22.14. Two copies of $\mathbb{R}_{\text{cocountable}}$ give a space that is locally connected but not connected in the other three ways.

23. Mon, Oct. 20

The next topic is one of the major ones in the course: compactness. As we will see, this is the analogue of a "closed and bounded subset" in a general space. The definition relies on the idea of coverings.

Definition 23.1. An **open cover** of X is a collection $\mathcal{U}$ of open subsets that cover X. In other words, $\bigcup_{U \in \mathcal{U}} U = X$. Given two covers $\mathcal{U}$ and $\mathcal{V}$ of X, we say that $\mathcal{V}$ is a **subcover** if $\mathcal{V} \subseteq \mathcal{U}$.

Definition 23.2. A space X is said to be **compact** if every open cover has a *finite* subcover (i.e. a cover involving finitely many open sets).

Example 23.3. Clearly any finite topological space is compact, no matter the topology.

Example 23.4. An infinite set with the discrete topology is *not* compact, as the collection of singletons gives an open cover with no finite subcover.

Example 23.5. $\mathbb{R}$ is not compact, as the open cover $\mathcal{U} = \{(-n, n) \mid n \in \mathbb{N}\}$ has no finite subcover.

Example 23.6. Similarly $[0, \infty)$ is not compact, as the open cover $\mathcal{U} = \{[0, n)\}$ has no finite subcover. Recall that $[0, \infty) \cong [a, b)$.

Theorem 23.7. *Let $a < b$. Then $[a, b]$ is a compact subset of $\mathbb{R}$.*

Proof. Let $\mathcal{U}$ be an open cover. Then some element of the cover must contain a. Pick such an element and call it U_1.

Consider the set

$$\mathcal{E} = \{c \in [a, b] \mid [a, c] \text{ is finitely covered by } \mathcal{U}\}.$$

Certainly $a \in \mathcal{E}$ and $\mathcal{E}$ is bounded above by b. By the Least Upper Bound Axiom, $s = \sup \mathcal{E}$ exists. Note that $a \leq s \leq b$, so we must have $s \in U_s$ for some $U_s \in \mathcal{U}$. But then for any $c < s$ with $c \in U_s$, we have $c \in \mathcal{E}$. This means that

$$[a, c] \subseteq U_1 \cup \cdots \cup U_k$$

for $U_1, \ldots, U_k \in \mathcal{U}$. But then $[a, s] \subseteq U_1 \cup \cdots \cup U_k \cup U_s$. This shows that $s \in \mathcal{E}$. On the other hand, the same argument shows that for any $s < d < b$ with $d \in U_s$, we would similarly have $d \in \mathcal{E}$. Since $s = \sup \mathcal{E}$, there cannot exist such a d. This implies that $s = b$. ■

Like connectedness, compactness is preserved by continuous functions.

Proposition 23.8. *Let $f : X \longrightarrow Y$ be continuous, and assume that X is compact. Then $f(X)$ is compact.*

Proof. Let $\mathcal{V}$ be an open cover of $f(X)$. Then $\mathcal{U} = \{f^{-1}(V) \mid V \in \mathcal{V}\}$ is an open cover of X. Let $\{U_1, \ldots, U_k\}$ be a finite subcover. It follows that the corresponding $\{V_1, \ldots, V_k\}$ is a finite subcover of $\mathcal{V}$. ■

Example 23.9. Recall that we have the quotient map $\exp : [0, 1] \longrightarrow S^1$. It follows that S^1 is compact.

Theorem 23.10 (Extreme Value Theorem)**.** *Let $f : [a, b] \longrightarrow \mathbb{R}$ be continuous. Then f attains a maximum and a minimum.*

Proof. Since f is continuous and $[a, b]$ is both connected and compact, the same must be true of its image. But the compact, connected subsets are precisely the closed intervals. ■

The following result is also quite useful.

Proposition 23.11. *Let X be Hausdorff and let $A \subseteq X$ be a compact subset. Then A is closed in X.*

Proof. Pick any point $x \in X \setminus A$ (if we can't, then $A = X$ and we are done). For each $a \in A$, we have disjoint neighborhoods $a \in U_a$ and $x \in V_a$. Since the U_a cover A, we only need finitely many, say $U_{a_1}, \dots, U_{a_k}$ to cover A. But then the intersection

$$V = V_{a_1} \cap \cdots \cap V_{a_k}$$

of the corresponding V_a's is disjoint from the union of the U_a's and therefore also from A. Since V is a finite intersection of open sets, it is open and thus gives a neighborhood of x in $X \setminus A$. It follows that A is closed. ■

Exercise 23.12. *If $A \subseteq X$ is closed and X is compact, then A is compact.*

Combining these results gives the following long-awaited consequence.

Corollary 23.13. *Let $f : X \longrightarrow Y$ be continuous, where X is compact and Y is Hausdorff, then f is a closed map.*

24. Wed, Oct. 22

In particular, if f is already known to be a continuous bijection, then it is automatically a homeomorphism. For example, this shows that the map $I/\partial I \longrightarrow S^1$ is a homeomorphism. Similarly, from Example 15.5 we have $D^n/\partial D^n \cong S^n$.

We will next show that finite products of compact spaces are compact, but we first need a lemma.

Lemma 24.1 (Tube Lemma)**.** *Let X be compact and Y be any space. If $W \subseteq X \times Y$ is open and contains $X \times \{y\}$, then there is a neighborhood V of y with $X \times V \subseteq W$.*

Proof. For each $x \in X$, we can find a basic neighborhood $U_x \times V_x$ of (x, y) in W. The U_x's give an open cover of X, so we only need finitely many of them, say $U_{x_1}, \dots, U_{x_n}$. Then we may take $V = V_{x_1} \cap \cdots \cap V_{x_n}$. ■

Proposition 24.2. *Let X and Y be nonempty. Then $X \times Y$ is compact if and only if X and Y are compact.*

Proof. As for connectedness, the continuous projections make X and Y compact if $X \times Y$ is compact.

Now suppose that X and Y are compact and let $\mathcal{U}$ be an open cover. For each $y \in Y$, the cover $\mathcal{U}$ of $X \times Y$ certainly covers the slice $X \times \{y\}$. This slice is homeomorphic to X and therefore finitely-covered by some $\mathcal{V} \subset \mathcal{U}$. By the Tube Lemma, there is a neighborhood V_y of y such that the tube $X \times V_y$ is covered by the same $\mathcal{V}$. Now the V_y's cover Y, so we only need finitely many of these to cover X. Since each tube is finitely covered by $\mathcal{U}$ and we can cover $X \times Y$ by finitely many tubes, it follows that $\mathcal{U}$ has a finite subcover. ■

Theorem 24.3 (Heine-Borel)**.** *A subset $A \subseteq \mathbb{R}^n$ is compact if and only if it is closed and bounded (contained in a single metric ball).*

Proof. Suppose A is compact. Then A must be closed in $\mathbb{R}^n$ since $\mathbb{R}^n$ is Hausdorff. To see that A is bounded, pick any point $a \in A$ (if A is empty, we are certainly done). Then the collection of balls $B_n(a) \cap A$ gives an open cover of A, since any other point in A is a finite distance away from a. Since A is compact, there must be a finite subcover $\{B_{n_1}(a), \dots, B_{n_k}(a)\}$. Let $N = \max\{n_1, \dots, n_k\}$. Then $A \subseteq B_N(a)$.

On the other hand, suppose that A is closed and bounded in $\mathbb{R}^n$. Since A is bounded, it is contained in $[-k, k]^n$ for some $k > 0$. But this product of intervals is compact since each interval is compact. Now A is a closed subset of a compact space, so it is compact. ■

In fact, the forward implication of the above proof works to show that

Proposition 24.4. *Let $A \subseteq X$, where X is metric and A is compact. Then A is closed and bounded in X.*

But the reverse implication is not true in general, as the next example shows.

Example 24.5. Consider $[0,1] \cap \mathbb{Q} \subseteq \mathbb{Q}$. This is certainly closed and bounded, but we will see it is not compact. Consider the open cover $\mathcal{U} = \big\{[0, \frac{1}{\pi} - \frac{1}{n})\big\}_{n\in\mathbb{N}} \cup \{(\frac{1}{\pi}, 1]\}$.

Again, we have shown that compactness interacts well with finite products, and we would like a similar result in the arbitrary product case. This is a major theorem, known as the Tychonoff theorem. First, we show the theorem does not hold with the box topology.

Example 24.6. Let $D = [-1, 1]$ and consider $D^{\mathbb{N}}$, equipped with the box topology. For each k, let introduce $L = [0, 1)$ and $R = (0, 1]$. Take cover by products of L's and R's. No finite subcover.

Theorem 24.7 (Tychonoff). *Let $X_i \neq \emptyset$ for all $i \in \mathcal{I}$. Then $\prod_i X_i$ is compact if and only if each X_i is compact.*

We will prove this next time. Our proof, even for the difficult direction, will use the axiom of choice. In fact, Tychonoff's theorem is equivalent to the axiom of choice.

Theorem 24.8. *Tychonoff $\Rightarrow$ axiom of choice.*

Proof. This argument is quite a bit simplier than the other implication. Let $X_i \neq \emptyset$ for all $i \in \mathcal{I}$. We want to show that $X = \prod_i X_i \neq \emptyset$.

For each i, define $Y_i = X_i \cup \{\infty_i\}$, where $\infty_i \notin X_i$. We topologize Y_i such that the only nontrivial open sets are X_i and $\{\infty_i\}$. Now for each i, let $U_i = p_i^{-1}(\infty_i)$. The collection $\mathcal{U} = \{U_i\}$ gives a collection of open subsets of $Y = \prod_i Y_i$, and this collection covers Y if and only if $X = \emptyset$. Each Y_i is compact since it has only four open sets. Thus Y must be compact by the Tychonoff theorem. But no finite subcollection of $\mathcal{U}$ can cover Y. For example, $U_i \cup U_j$ does not cover Y since if $a \in X_i$ and $b \in X_j$, then we can define $(y_i) \in Y \setminus (U_i \cup U_j)$ by

$$y_k = \begin{cases} a & k = i \\ b & k = j \\ \infty_k & k \neq i, j \end{cases}$$

The same kind of argument will work for any finite collection of U_i's. Since $\mathcal{U}$ has no finite subcover and Y is compact, $\mathcal{U}$ cannot cover Y, so that X must be nonempty. ■

25. Fri, Oct. 24

Started by correcting Example 24.6.

Here is a simpler example of a noncompact product in the box topology. Consider $\{0,1\}^{\mathbb{N}}$. In the box topology, this space is discrete. Since it is infinite, it is not compact.

It turns out that the Tychonoff Theorem is *equivalent* to the axiom of choice. We will thus use a form of the axiom of choice in order to prove it.

Zorn's Lemma. Let P be a partially-ordered set. If every linearly-ordered subset of P has an upper bound in P, then P contains at least one maximal element.

Theorem 25.1 (Tychonoff). *Let $X_i \neq \emptyset$ for all $i \in \mathcal{I}$. Then $\prod_i X_i$ is compact if and only if each X_i is compact.*

Proof. As we have seen a number of times, the implication ($\Rightarrow$) is trivial.

We now show the contrapositive of ($\Leftarrow$). Thus assume that $X = \prod_i X_i$ is not compact. We wish to conclude that one of the X_i must be noncompact. By hypothesis, there exists an open cover $\mathcal{U}$ of X with no finite subcover.

We first deal with the following case.

Special case: *$\mathcal{U}$ is a cover by prebasis elements.*
For each $i \in \mathcal{I}$, let $\mathcal{U}_i$ be the collection

$$\mathcal{U}_i = \{V \subseteq X_i \text{ open } \mid p_i^{-1}(V) \in \mathcal{U}\}.$$

For some i, the collection $\mathcal{U}_i$ must cover X_i, since otherwise we could pick $x_i \in X_i$ for each i with x_i not in the union of $\mathcal{U}_i$. Then the element $(x_i) \in \prod_i X_i$ would not be in $\mathcal{U}$ since it cannot be in any $p_i^{-1}(V)$. Then $\mathcal{U}_i$ cannot have a finite subcover, since that would provide a corresponding subcover of $\mathcal{U}$. It follows that X_i is not compact.

It remains to show that we can always reduce to the cover-by-prebasis case.

Consider the collection $\mathcal{N}$ of open covers of X having no finite subcovers. By assumption, this set is nonempty, and we can partially order $\mathcal{N}$ by inclusion of covers. Furthermore, if $\{\mathcal{U}_\alpha\}$ is a linearly order subset of $\mathcal{N}$, then $\mathcal{U} = \bigcup_\alpha \mathcal{U}_\alpha$ is an open cover, and it cannot have a finite subcover since a finite subcover of $\mathcal{U}$ would be a finite subcover of one of the $\mathcal{U}_\alpha$. Thus $\mathcal{U}$ is an upper bound in $\mathcal{N}$ for $\{\mathcal{U}_\alpha\}$. By Zorn's Lemma, $\mathcal{N}$ has a maximal element $\mathcal{V}$.

Now let $\mathcal{S} \subseteq \mathcal{V}$ be the subcollection consisting of the prebasis elements in $\mathcal{V}$. We claim that $\mathcal{S}$ covers X. *Suppose not.* Thus let $x \in X$ such that x is not covered by $\mathcal{S}$. Then x must be in V for some $V \in \mathcal{V}$. By the definition of the product topology, x must have a basic open neighborhood in $B \subset V$. But any basic open set is a finite intersection of prebasic open sets, so $B = S_1 \cap \dots S_k$. If x is not covered by $\mathcal{S}$, then none of the S_i are in $\mathcal{S}$. Thus $\mathcal{V} \cup \{S_i\}$ is not in $\mathcal{N}$ by maximality of $\mathcal{V}$. In other words, $\mathcal{V} \cup \{S_i\}$ has a finite subcover $\{V_{i,1}, \dots, V_{i,n_i}, S_i\}$. Let us write

$$\hat{V}_i = V_{i,1} \cup \dots \cup V_{i,n_i}.$$

Now

$$X = \bigcap_i \left(S_i \cup \hat{V}_i\right) \subseteq \left(\bigcap_i S_i\right) \cup \left(\bigcup_i \hat{V}_i\right) \subseteq V \cup \left(\bigcup_i \hat{V}_i\right)$$

This shows that $\mathcal{V}$ has a finite subcover, which contradicts that $\mathcal{V} \in \mathcal{N}$. We thus conclude that $\mathcal{S}$ covers X using only prebasis elements.

But now by the argument at the beginning of the proof, $\mathcal{S}$, and therefore $\mathcal{V}$ as well, has a finite subcover. This is a contradiction. ∎

Closely related to compactness is the following notion.

Definition 26.1. We say that a space X is **sequentially compact** if every sequence in X has a convergent subsquence.

Example 26.2. The open interval $(0, 1)$ is not sequentially compact because $\{1/n\}$ has no subsequence that converges in $(0, 1)$. If we consider instead $[0, 1]$, this example no longer works, and we will see that $[0, 1]$ is indeed sequentially compact.

In general, there is no direct relation between compactness and sequential compactness.

Example 26.3. Consider $X = \prod_{[0,1]}\{0, 1\}$ under the product topology. By the Tychonoff theorem, X is compact. However, it is not sequentially compact. Let $f_n \in X$ be defined by $f_n(x) =$ the nth digit in the binary expansion of x. We claim that (f_n) has no convergent subsequence. Recall that convergence in X means *pointwise* convergence of functions. Let (f_{n_k}) be any subsequence. In order for this to converge, it the sequence $f_{n_k}(x)$ would need to converge for every x. This is simply a sequence of 0's and 1's, so it must be eventually constant. But no matter the subsequence f_{n_k}, we can find an $x \in I$ whose corresponding sequence of digits is not eventually constant.

Example 26.4. Let

$$X = \left\{ x \in \prod_{\mathbb{R}}\{0, 1\} \,\middle|\, x^{-1}(1) \text{ is countable.} \right\}$$

We here consider $\{0, 1\}$ with the discrete topology, and X is a subspace of the product. For each $r \in \mathbb{R}$, let $B_r = \{x \in X \mid x(r) = 0\}$. This is a prebasis element and so is open. Then the collection $\{B_r\}_{r\in\mathbb{R}}$ gives an open cover of X, but it clearly has no finite subcover.

Now let (x_n) be a sequence in X. Let

$$S = \bigcup_n x_n^{-1}(1).$$

S is a countable union of countable sets, so it is countable. Let $Y = \prod_S\{0, 1\}$, and let $q : X \longrightarrow Y$ be the restriction along $S \hookrightarrow \mathbb{R}$. Then $q(x_n)$ is a sequence in $Y = \{0, 1\}^S$. It can be seen directly that Y is sequentially compact, so that some subsequence $q(x_{n_k})$ of $q(x_n)$ must converge to, say $y \in Y$. Let $z \in X$ be the function with $x^{-1}(1) = y^{-1}(1)$. But then x_{n_k} converges to z since each x_n is identically 0 on $\mathbb{R} \setminus S$.

We have shown that X is sequentially compact space but not compact.

There is one more form of compactness.

Definition 26.5. A space is said to be **limit point compact** of every infinite subset has a limit point (accumulation point).

Theorem 26.6. *If X is a metric space, then X is compact if and only if it is sequentually compact if and only if it is limit point compact.*

Proof. See Munkres, Theorem 28.2 or Lee, Lemmas 4.42-4.44. ■

In $\mathbb{R}^n$, this result is knwon by the following name.

Theorem 26.7 (Bolzano-Weierstrass)**.** *Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.*

We also saw that the compact subsets of the metric space $\mathbb{R}^n$ are the closed and bounded ones. Do we have an analogue of this statement for an arbitrary metric space X? First, note that closed and bounded is not enough in general to guarantee compactness, as any infinite discrete metric space shows.

27. Wed, Oct. 29

We discussed the takehome exam today.

28. Fri, Oct. 31

Definition 28.1. We say that a metric space X is **totally bounded** if, for every $\epsilon > 0$, there is a finite covering of X by ϵ-balls.

It is clear that compact implies totally bounded because, for any fixed $\epsilon > 0$, the B_ϵ give an open covering. This suffices to handle the discrete metric case, as a discrete metric space is totally bounded $\iff$ it is finite $\iff$ it is compact. However, closed and totally bounded is still not enough, as $[0,1] \cap \mathbb{Q}$ is closed and totally bounded (either in $\mathbb{Q}$ or in itself) but not compact, as we have already seen. Recall that a metric space is **complete** if every Cauchy sequence converges in X.

Theorem 28.2. *Let X be metric. Then X is compact $\iff$ X is complete and totally bounded.*

Proof. ($\Rightarrow$) We have already mentioned why compactness implies totally bounded. Let (x_n) be a Cauchy sequence in X. Then, since X is sequentially compact, a subsequence of (x_n) converges. But if $x_{n_k} \to x$, then we must also have $x_n \to x$ since x_n is Cauchy (prove this)! It follows that X is complete.

($\Leftarrow$) Suppose now that X is complete and totally bounded. We show that X is sequentially compact. Let (x_n) be any sequence in X. Since X is complete, it suffices to show that (x_n) has a subsequence that is Cauchy.

For each n, we have a finite covering of X by k_n balls of radius $1/n$. Start with $n = 1$. One of these balls must contain infinitely many x_n's and so a subsequence of (x_n). Now cover X by finitely many balls of radius $1/2$. Again, one of these contains a subsequence of the previous subsequence. We continue in this way ad infinitum. We obtain the desired Cauchy subsequence as follows. First, pick x_{n_1} to be in our original subsequence (in the chosen ball of radius 1). Then pick x_{n_2} to be in the subsubsequence in our chosen ball of radius $1/2$ (and pick it such that $n_2 > n_1$. After (many, many) choices, we get a subsequence of x_n such that $\{x_{n_k}\}_{k \geq m}$ is contained in a ball of radius $1/m$. It follows that x_{n_k} is Cauchy. ∎

Note that $[0,1] \cap \mathbb{Q}$ is not complete, as the sequence

$$x_n = \text{ the decimal expansion of } 1/\pi \text{ cut off after the } n\text{th digit}$$

is a Cauchy sequence in $[0,1] \cap \mathbb{Q}$ which does not converge.

Definition 28.3. We say that a space is **locally compact** if every $x \in X$ has a compact neighborhood (recall that we do not require neighborhoods to be open).

This looks different from our other "local" notions. To get a statement in the form we expect, we introduce more terminology $A \subseteq X$ is **precompact** if $\overline{A}$ is compact.

Proposition 28.4. *Let X be Hausdorff. TFAE*

(1) *X is locally compact*
(2) *every $x \in X$ has a precompact neighborhood*
(3) *X has a basis of precompact open sets*

Proof. It is clear that (3) $\Rightarrow$ (2) $\Rightarrow$ (1) without the the Hausdorff assumption, so we show that (1) $\Rightarrow$ (3). Suppose X is locally compact and Hausdorff. Let V be open in X and let $x \in V$. We want a precompact open neighborhood of x in V. Since X is locally compact, we have a compact neighborhood K of x, and since X is Hausdorff, K must be closed. Since V and K are both neighborhoods of x, so is $V \cap K$. Thus let $x \in U \subseteq V \cap K$. Then $\overline{U} \subseteq K$ since K is closed, and $\overline{U}$ is compact since it is a closed subset of a compact set. ■

In contrast to the local connectivity properties, it is clear that any compact space is locally compact. But this is certainly a generalization of compactness, since *any interval* in $\mathbb{R}$ is locally compact.

Example 28.5. A standard example of a space that is not locally compact is $\mathbb{Q} \subseteq \mathbb{R}$. We show that 0 does not have any compact neighborhoods . Let V be any neighborhood of 0. Then it must contain $(-\pi/n, \pi/n)$ for some n. Now

$$\mathcal{U} = \left\{ \left(-\pi/n, \left(\frac{k}{k+1} \right) \pi/n \right) \right\} \cup \left\{ V \cap (\pi/n, \infty), V \cap (-\infty, -\pi/n) \right\}$$

is an open cover of V with no finite subcover.

Remark 28.6. Why did we define local compactness in a different way from local (path)-connectedness? We could have defined locally connected to mean that every point has a connected neighborhood , which follows from the actual definition. But then we would not have that locally connected is equivalent to having a basis of connected open sets. On the other hand, we could try the $x \in K \subseteq U$ version of locally compact, but of course we don't want to allow $K = \{x\}$, so the next thing to require is $x \in V \subseteq U$, where V is precompact. As we showed in Prop 28.4, this is equivalent to our definition of locally compact in the presence of the Hausdorff condition. Without the Hausdorff condition, compactness does not behave quite how we expect.

29. Mon, Nov. 3

Locally compact Hausdorff spaces are a very nice class of spaces (almost as good as compact Hausdorff). In fact, any such space is close to a compact Hausdorff space.

Definition 29.1. A **compactification** of a noncompact space X is an embedding $i : X \hookrightarrow Y$, where Y is compact and $i(X)$ is dense.

We will typically work with Hausdorff spaces X, in which case we ask the compactification Y to also be Hausdorff.

Example 29.2. The open interval $(0,1)$ is not compact, but $(0,1) \hookrightarrow [0,1]$ is a compactification. Note that the exponential map $\exp : (0,1) \longrightarrow S^1$ also gives a (different) compactification.

There is often a smallest compactification, given by the following construction.

Definition 29.3. Let X be a space and define $\widehat{X} = X \cup \{\infty\}$, where $U \subseteq \widehat{X}$ is open if either

- $U \subseteq X$ and U is open in X or
- $\infty \in U$ and $\widehat{X} \setminus U \subseteq X$ is compact.

Proposition 29.4. *Suppose that X is Hausdorff and noncompact. Then $\widehat{X}$ is a compactification. If X is locally compact, then $\widehat{X}$ is Hausdorff.*

Proof. We first show that $\widehat{X}$ is a space! It is clear that both $\emptyset$ and $\widehat{X}$ are open.

Suppose that U_1 and U_2 are open. We wish to show that $U_1 \cap U_2$ is open.

- If neither open set contains ∞, this follows since X is a space.
- If $\infty \in U_1$ but $\infty \notin U_2$, then $K_1 = X \setminus U_1$ is compact. Since X is Hausdorff, K_1 is closed in X. Thus $X \setminus K_1 = U_1 \setminus \{\infty\}$ is open in X, and it follows that $U_1 \cap U_2 = (U_1 \setminus \{\infty\}) \cap U_2$ is open.
- If $\infty \in U_1 \cap U_2$, then $K_1 = X \setminus U_1$ and $K_2 = X \setminus U_2$ are compact. It follows that $K_1 \cup K_2$ is compact, so that $U_1 \cap U_2 = X \setminus (K_1 \cup K_2)$ is open.
- Suppose we have a collection U_i of open sets. If none contain ∞, then neither does $\bigcup_i U_i$, and the union is open in X. If $\infty \in U_j$ for some j, then $\infty \in \bigcup_i U_i$ and
$$\widehat{X} \setminus \bigcup_i U_i = \bigcap_i (\widehat{X} \setminus U_i) = \bigcap_i (X \setminus U_i)$$
is a closed subset of the compact set $X \setminus U_j$, so it must be compact.

Next, we show that $\iota : X \longrightarrow \widehat{X}$ is an embedding. Continuity of ι again uses that compact subsets of X are closed. That ι is open follows immediately from the definition of $\widehat{X}$.

To see that $\iota(X)$ is dense in $\widehat{X}$, it suffices to see that $\{\infty\}$ is not open. But this follows from the definition of $\widehat{X}$, since X is not compact.

Finally, we show that $\widehat{X}$ is compact. Let $\mathcal{U}$ be an open cover. Then some $U \in \mathcal{U}$ must contain ∞. The remaining elements of $\mathcal{U}$ must cover $X \setminus U$, which is compact. It follows that we can cover $X \setminus U$ using only finitely many elements, so $\mathcal{U}$ has a finite subcover.

Now suppose that X is locally compact. Let x_1 and x_2 in $\widehat{X}$. If neither is ∞, then we have disjoint neighborhoods in X, and these are still disjoint neighborhoods in $\widehat{X}$. If $x_2 = \infty$, let $x_1 \in U \subseteq K$, where U is open and K is compact. Then U and $V = \widehat{X} \setminus K$ are the desired disjoint neighborhoods
. ■

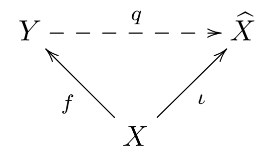
Example 29.5. We saw that S^1 is a one-point compactification of $(0,1) \cong \mathbb{R}$. You will show on your homework that similarly S^n is a one-point compactification of $\mathbb{R}^n$.

Example 29.6. As we have seen, $\mathbb{Q}$ is not locally compact, so we do not expect $\widehat{\mathbb{Q}}$ to be Hausdorff. Indeed, the point ∞ is dense in $\widehat{\mathbb{Q}}$. Because of the topology on $\widehat{\mathbb{Q}}$, this is equivalent to showing that for any open, nonempty subset $U \subseteq \mathbb{Q}$, U is not contained in any compact subset. Since $\mathbb{Q}$ is Hausdorff, if U were contained in a compact subset, then $\overline{U}$ would also be compact. But as we have seen, for any interval $(a,b) \cap \mathbb{Q}$, the closure in $\mathbb{Q}$, which is $[a,b] \cap \mathbb{Q}$, is not compact.

30. Wed, Nov. 5

Next, we show that the situation we observed for compactifications of $(0,1)$ holds quite generally.

Proposition 30.1. *Let X be locally compact Hausdorff and let $f : X \longrightarrow Y$ be a compactification. Then there is a (unique) quotient map $q : Y \longrightarrow \widehat{X}$ such that $q \circ f = \iota$.*



We will need:

Lemma 30.2. *Let X be locally compact Hausdorff and $f : X \longrightarrow Y$ a compactification. Then f is open.*

Proof. Since f is an emebedding, we can pretend that $X \subseteq Y$ and that f is simply the inclusion. We wish to show that X is open in Y. Thus let $x \in X$. Let U be a precompact neighborhood of x. Thus $K = \mathrm{cl}_X(U)$ is compact[3] and so must be closed in Y (and X) since Y is Hausdorff. By the definition of the subspace topology, we must have $U = V \cap X$ for some open $V \subseteq Y$. Then V is a neighborhood of x in Y, and

$$V = V \cap Y = V \cap \mathrm{cl}_Y(X) \subseteq \mathrm{cl}_Y(V \cap X) = K \subseteq X.$$

■

Proof of Prop. 30.1. We define

$$q(y) = \begin{cases} \iota(x) & \text{if } y = f(x) \\ \infty & \text{if } y \notin f(X). \end{cases}$$

To see that q is continuous, let $U \subseteq \widehat{X}$ be open. If $\infty \notin U$, then $q^{-1}(U) = f(\iota^{-1}(U))$ is open by the lemma. If $\infty \in U$, then $K = \widehat{X} \setminus U$ is compact and thus closed. We have $q^{-1}(K) = f(\iota^{-1}(K))$ is compact and closed in Y, so it follows that $q^{-1}(U) = Y \setminus q^{-1}(K)$ is open.

Note that q is automatically a quotient map since it is a closed continuous surjection (it is closed because Y is compact and $\widehat{X}$ is Hausdorff). Note also that q is unique because $\widehat{X}$ is Hausdorff and q is already specified on the dense subset $f(X) \subseteq Y$. ■

Remark 30.3. Note that if we apply the one-point compactification to a (locally compact) metric space X, there is no natural metric to put on X, so one might ask for a good notion of compactification for metric spaces. Given the result above, this should be related to the idea of a completion of a metric space. See HW8.

The following result is often useful, and it matches more closely what we might have expected the definition of locally compact to resemble.

Proposition 30.4. *Let X be locally compact and Hausdorff. Let U be an open neighborhood of x. Then there is a precompact open set V with*

$$x \in V \subseteq \overline{V} \subseteq U.$$

---

[3] We will need to distinguish between closures in X and closures in Y, so we use the notation $\mathrm{cl}_X(A)$ for closure rather than our usual $\overline{A}$.

Proof. We use Prop 29.4. Thus let $X \hookrightarrow Y$ be the one-point compactification. By definition, U is still open in Y, so $K = Y - U$ is closed in Y and therefore compact. By HW 7.3, we can find disjoint open sets V and W in Y with $x \in V$ and $K \subseteq W$. Since W is open, it follows that $\overline{V}$ is disjoint from W and therefore also from K. In other words, $\overline{V}$ is contained in U. ■

Proposition 30.5. *A space X is Hausdorff and locally compact if and only if it is homeomorphic to an open subset of a compact Hausdorff space Y.*

Proof. ($\Rightarrow$). We saw that X is open in the compact Hausdorff space $Y = \widehat{X}$.

($\Leftarrow$) As a subspace of a Hausdorff space, it is clear that X is Hausdorff. It remains to show that every point has a compact neighborhood (in X). Write $Y_\infty = Y \setminus X$. This is closed in Y and therefore compact. By Problem 3 from HW7, we can find disjoint open sets $x \in U$ and $Y_\infty \subseteq V$ in Y. Then $K = Y \setminus V$ is the desired compact neighborhood of x in X. ■

Corollary 30.6. *If X and Y are locally compact Hausdorff, then so is $X \times Y$.*

Corollary 30.7. *Any open or closed subset of a locally compact Hausdorff space is locally compact Hausdorff.*

31. Fri, Nov 7

We finally turn to the so-called "separation axioms".

Definition 31.1. A space X is said to be

- T_0 if given two distinct points x and y, there is a neighborhood of one not containing the other
- T_1 if given two distinct points x and y, there is a neighborhood of x not containing y and vice versa (points are closed)
- T_2 (**Hausdorff**) if any two distinct points x and y have disjoint neighborhoods
- T_3 (**regular**) if *points are closed and* given a closed subset A and $x \notin A$, there are disjoint open sets U and V with $A \subseteq U$ and $x \in V$
- T_4 (**normal**) if *points are closed and* given closed disjoint subsets A and B, there are disjoint open sets U and V with $A \subseteq U$ and $B \subseteq V$.

Note that $T_4 \implies T_3 \implies T_2 \implies T_1 \implies T_0$. But beware that in some literature, the "points are closed" clause is not included in the definition of regular or normal. Without that, we would not be able to deduce T_2 from T_3 or T_4.

We have talked a lot about Hausdorff spaces. The other important separation property is T_4. We will not really discuss the intermediate notion of regular (or the other variants completely regular, completely normal, etc.)

Proposition 31.2. *Any compact Hausdorff space is normal.*

Proof. This was homework problem 7.3. ■

Later in the course, we will see that this generalizes to locally compact Hausdorff, as long as we add in the assumption that the space is second-countable. Another important class of normal spaces is the collection of metric spaces.

Proposition 31.3. *If X is metric, then it is normal.*

Proof. Let X be metric and let $A, B \subseteq X$ be closed and disjoint. For every $a \in A$, let $\epsilon_a > 0$ be a number such that $B_{\epsilon_a}(a)$ does not meet B (using that B is closed). Let

$$U_A = \bigcup_{a \in A} B_{\epsilon_a/2}(a).$$

Similarly, we let

$$U_B = \bigcup_{b \in B} B_{\epsilon_b/2}(b).$$

It only remains to show that U_A and U_B must be disjoint. Let $x \in B_{\epsilon_a/2}(a) \subseteq U_A$ and pick any $b \in B$. We have

$$d(a,x) < \frac{1}{2}\epsilon_a < \frac{1}{2}d(a,b)$$

and thus

$$d(x,b) \geq d(a,b) - d(a,x) > d(a,b) - \frac{1}{2}d(a,b) = \frac{1}{2}d(a,b) > \frac{1}{2}\epsilon_b.$$

It follows that $U_A \cap U_B = \emptyset$. ■

Unfortunately, the T_4 condition alone is not preserved by the constructions we have studied.

Example 31.4. (Images) We will see that $\mathbb{R}$ is normal. But recall the quotient map $q : \mathbb{R} \longrightarrow \{-1, 0, 1\}$ which sends any number to its sign. This quotient is not Hausdorff and therefore not (regular or) normal.

Example 31.5. (Subspaces) If J is uncountable, then the product $(0,1)^J$ is *not* normal (Munkres, example 32.2). This is a subspace of $[0,1]^J$, which is compact Hausdorff by the Tychonoff theorem and therefore normal. So a subspace of a normal space need not be normal. We also saw in this example that (uncountable) products of normal spaces need not be normal.

Example 31.6. (Products) The lower limit topology $\mathbb{R}_{\ell\ell}$ is normal (Munkres, example 31.2), but $\mathbb{R}_{\ell\ell} \times \mathbb{R}_{\ell\ell}$ is *not* normal (Munkres, example 31.3). Note that this also gives an example of a Hausdorff space that is not normal.

Ok, so we've seen a few examples. So what, why should we care about normal spaces? Look back at the definition for T_2, T_3, T_4. In each case, we need to find certain open sets U and V. How would one do this in general? In a metric space, we would build these up by taking unions of balls. In an arbitrary space, we might use a basis. But another way of getting open sets is by pulling back open sets under a continuous map. That is, suppose we have a map $f : X \longrightarrow [0,1]$ such that $f \equiv 0$ on A and $f \equiv 1$ on B. Then $A \subseteq U := f^{-1}([0, \frac{1}{2}))$ and $B \subseteq V := f^{-1}((\frac{1}{2}, 1])$, and $U \cap V =$.

First, note that the definition of normal, by considering the complement of B, can be restated as

Lemma 31.7. *Let X be normal, and suppose given $A \subseteq U$ with A closed and U open. Then there exists an open set V with*

$$A \subseteq V \subseteq \overline{V} \subseteq U.$$

Now we have another very important result.

Theorem 31.8 (Urysohn's Lemma)**.** *Let X be normal and let A and B be disjoint closed subsets. Then there exists a continuous function $f : X \longrightarrow [0,1]$ such that $A \subseteq f^{-1}(0)$ and $B \subseteq f^{-1}(1)$.*

Sketch of proof. Define $U_1 = X \setminus B$, so that we have $A \subseteq U_1$. Since X is normal, we can find an open U_0 with $A \subseteq U_0 \subseteq \overline{U_0} \subseteq U_1$. By induction on the rational numbers $r \in \mathbb{Q} \cap (0,1)$, we can find for each r an open set U_r with $\overline{U_r} \subset U_s$ if $r < s$. We also define $U_r = X$ for $r > 1$. Then define

$$f(x) = \inf\{r \in \mathbb{Q} \cap [0, 1.001) \mid x \in U_r\}.$$

Now if $x \in A$, then $x \in U_0$, so $f(x) = 0$ as desired. If $x \in B$, then $x \notin U_1$, but $x \in U_r$ for any $r > 1$, so $f(x) = 1$ as desired. It remains to show that f is continuous.

It suffices to show that the preimage under f of the prebasis elements $(-\infty, a)$ and (a, ∞) are open. We have

$$f^{-1}(-\infty, a) = \bigcup_{\substack{r \in \mathbb{Q} \\ r < a}} U_r, \qquad \text{and} \qquad f^{-1}(a, \infty) = \bigcup_{\substack{r \in Q \\ r > a}} X \setminus \overline{U_r}$$

To see the second equality, note that if $f(x) > a$ then for any $a < r < f(x)$, we have $x \notin U_r$. But we can then find $r < s < f(x)$, so that $x \notin U_s \supseteq \overline{U_r} \supseteq U_r$. For more details, see either [Lee, Thm 4.82] or [Munkres, Thm 33.1]. ■

Note that Urysohn's Lemma becomes an if and only if statement if we either drop the T_1-condition from normal or if we explicitly include singletons as possible replacements for A and B.

32. Mon, Nov. 10

Last time, we saw that a space is normal if and only if any two closed sets can be separated by a continuous function (modulo the T_1 condition). Here is another important application of normal spaces.

Theorem 32.1 (Tietze extension theorem)**.** *Suppose X is normal and $A \subseteq X$ is closed. Then any continuous function $f : A \longrightarrow [0,1]$ can be extended to a continuous function $\tilde{f} : X \longrightarrow [0,1]$.*

Again, this becomes an if and only if if we drop the T_1-condition from normal.

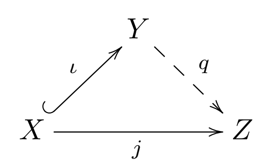
It is also easy to see that the result fails if we drop the hypothesis that A be closed. Consider $X = S^1$ and A is the complement of a point. Then we know that $A \cong (0,1)$, but this homeomorphism cannot extend to a map $S^1 \twoheadrightarrow (0,1)$.

Sketch of proof. It is more convenient for the purpose of the proof to work with the interval $[-1,1]$ rather than $[0,1]$. Thus suppose $f : A \longrightarrow [-1,1]$ is continuous. Then $A_1 = f^{-1}([-1,-1/3])$ and $B_1 = f^{-1}([1/3,1])$ are closed, disjoint subsets of A and therefore also of X. Since X is normal, we have a Urysohn function $g_1 : X \longrightarrow [-1/3,1/3]$ which separates A_1 and B_1. It is simple to check that $|f(a) - g_1(a)| \leq 2/3$ for all $a \in A$. In other words, we have a map

$$f_1 = f - g_1 : A \longrightarrow [-2/3,2/3].$$

Define $A_2 = f_1^{-1}([-2/3,-2/9])$ and $B_2 = f_1^{-1}([2/9,2/3])$. We get a Urysohn function $g_2 : X \longrightarrow [-2/9,2/9]$ which separates A_2 and B_2. Then the difference $f_2 = f - g_1 - g_2$ maps to $[-4/9,4/9]$. We continue in this way, and in the end, we get a sequence of functions (g_n) defined on X, and we define $g = \sum_n g_n$. By construction, this agrees with f on A (the difference will be less than $(2/3)^n$ for all n). The work remains in showing that the series defining g converges (compare to a geometric series) and that the resulting g is continuous (show that the series converges uniformly). See [Munkres, Thm 35.1] for more details. ■

Theorem 32.2 (Stone-Čech compactification)**.** *Suppose X is normal. There exists a "universal" compactification $\iota : X \longrightarrow Y$ of X, such that if $j : X \longrightarrow Z$ is any map to a compact Hausdorf space (for example a compactification), there is a unique map $q : Y \longrightarrow Z$ with $q \circ \iota = j$.*



33. Wed, Nov. 12

Proof. Given the space X, let $\mathcal{F} = \{ \text{ cts } f : X \longrightarrow [0,1]\}$. Define

$$\iota : X \longrightarrow [0,1]^{\mathcal{F}}$$

by $\iota(x)_f = f(x)$. This is continuous because each coordinate function is given by some $f \in \mathcal{F}$. The infinite cube is compact Hausdorff, and we let $Y = \overline{\iota(X)}$. It remains to show that ι is an embedding and also to demonstrate the universal property.

First, ι is injective by Urysohn's lemma: given distinct points x and y in X, there is a Urysohn function separating x and y, so $\iota(x) \neq \iota(y)$.

Now suppose that $U \subseteq X$ is open. We wish to show that $\iota(U)$ is open in $\iota(X)$. Pick $x_0 \in U$. Again by Urysohn's lemma, we have a function $g : X \longrightarrow [0,1]$ with $g(x_0) = 0$ and $g \equiv 1$ outside of U. Let

$$B = \{\iota(x) \in \iota(X) \mid g(x) \neq 1\} = \iota(X) \cap p_g^{-1}([0,1)).$$

Certainly $\iota(x_0) \in B$. Finally, $B \subset \iota(U)$ since if $\iota(x) \in B$, then $g(x) \neq 1$. But $g \equiv 1$ outside of U, so x must be in U.

For the universal property, suppose that $j : X \longrightarrow Z$ is a map to a compact Hausdorff space. Then Z is also normal, and the argument above shows that it embeds inside some large cube $[0,1]^K$. For each $k : Z \longrightarrow [0,1]$ in K, we thus get a coordinate map $i_k = p_k \circ j : X \longrightarrow [0,1]$, and it is clear how to extend this to get a map $q_k : Y \longrightarrow [0,1]$: just take q_k to be the projection map p_{i_k} onto the factor labelled by the map i_k. Piecing these together gives a map $q : Y \longrightarrow [0,1]^K$, but it restricts to the map j on the subset X. Since j has image in the closed subset Z, it follows that $q(Y) \subseteq Z$ since q is continuous and $\iota(X)$ is dense in Y. Note that q is the *unique* extension of j to Y since Z is Hausdorff and $\iota(X)$ is dense in Y. ■

Corollary 33.1. *Suppose that X is normal, and that $X \hookrightarrow Z$ is any compactification. Then Z is a quotient of the Stone-Čech compactification Y of X.*

Proof. According to the Theorem 32.2, we have a continuous map $q : Y \longrightarrow Z$ whose restriction to X is the given map $j : X \hookrightarrow Z$. The map q is closed since Y is compact and Z is Hausdorff. Also, $j(X)$ is dense in Z, and $j(X) = q(\iota(X)) \subseteq q(Y)$ so $q(Y) = Z$. In other words, q is closed, continuous, and surjective, therefore it is a quotient map. ■

The Stone-Čech compactification has consequences for *metrizability* of a space. Consider first the case that the index set J is countable.

Proposition 33.2. *Let Y be a metric space, and let $\overline{d} : Y \times Y \longrightarrow \mathbb{R}$ be the associated truncated metric. Then the formula*

$$D(\mathbf{y}, \mathbf{z}) = \sup \left\{ \frac{\overline{d}(y_n, z_n)}{n} \right\}$$

defines a metric on $Y^{\mathbb{N}}$, and the induced topology is the product topology.

Proof. We leave as an exercise the verification that this is a metric. We check the statement about the topology. For each n, let $p_n : Y^{\mathbb{N}} \longrightarrow Y$ be evaluation in the nth place. This is continuous, as given $\mathbf{y} \in Y^{\mathbb{N}}$ and $\epsilon > 0$, we let $\delta = \epsilon/n$. Then if $D(\mathbf{y}, \mathbf{z}) < \delta$, it follows that

$$d(y_n, z_n) = n \frac{d(y_n, z_n)}{n} \leq n D(\mathbf{y}, \mathbf{z}) < n\delta = \epsilon.$$

By the universal property of the product, we get a continuous bijection $p : Y^{\mathbb{N}} \longrightarrow \prod_{\mathbb{N}} Y$.

It remains to show that p is open. Thus let $B \subseteq Y^{\mathbb{N}}$ be an open ball, and let $\mathbf{y} \in p(B) = B$. We want to find a basis element U in the product topology with $\mathbf{y} \in U \subseteq B$. For convenience, we replace B by $B_\epsilon(\mathbf{y})$ for small enough ϵ. Take N large such that $1/N < \epsilon$. Then define

$$U = \bigcap_{i=1}^{N} p_i^{-1}(B_\epsilon(y_i)).$$

Let $\mathbf{z} \in Y^{\mathbb{N}}$. Recall that we have truncated our metric on Y at 1. Thus if $n > N$, we have that $\overline{d}(y_n, z_n)/n \leq 1/n \leq 1/N < \epsilon$. It follows that for any $\mathbf{z} \in U$, we have $\mathbf{z} \in B_\epsilon(\mathbf{x})$ as desired. ■

34. Fri, Nov. 14

On the other hand, if J is uncountable, then $[0,1]^J$ need not be metric, as the following example shows.

Example 34.1. The sequence lemma fails in $\mathbb{R}^{\mathbb{R}}$. Let $A \subseteq \mathbb{R}^{\mathbb{R}}$ be the subset consisting of functions that zero at all but finitely many points. Let g be the constant function at 1. Then $g \in \overline{A}$, since if

$$U = \bigcap_{x_1, \ldots, x_k} p_{x_i}^{-1}(a_i, b_i)$$

is a neighborhood of g, then the function

$$f(x) = \begin{cases} 1 & x \in \{x_1, \dots, x_k\} \\ 0 & \text{else} \end{cases}$$

is in $U \cap A$. But no sequence in A can converge to g (recall that convergence in the product topology means pointwise convergence). For suppose f_n is a sequence in A. For each n, let $Z_n = \mathrm{supp}(f_n)$ (the support is the set where f_n is nonzero). Then the set

$$\mathcal{Z} = \bigcup_n Z_n$$

is countable, and on the complement of $\mathcal{Z}$, all f_n's are zero. So it follows that the same must be true for any limit of f_n. Thus the f_n cannot converge to g.

This finally leads to a characterization of those topological spaces which come from metric spaces.

Theorem 34.2. *If X is normal and second countable, then it is metrizable.*

Proof. Since X is normal, we can embed X as above inside a cube $[0,1]^J$ for some J. Above, we took J to be the collection of all functions $X \longrightarrow [0,1]$.

To get a countable indexing set J, start with a countable basis $\mathcal{B} = \{B_n\}$ for X. For each pair of indices n, m for which $\overline{B}_n \subset B_m$, the Urysohn lemma gives us a function $g_{n,m}$ vanishing on $\overline{B}_n$ and equal to 1 outside B_m. We take $J = \{g_{n,m}\}$. Going back to the proof of the Stone-Čech-compactification, we needed, for any $x_0 \in X$ and $x_0 \in U$, to be able to find a function vanishing at x_0 but equal to 1 outside of U.

Take a basis element B_m satisfying $x_0 \in B_m \subset U$. Since X is normal, we can find an open set V with $x_0 \in V \subset \overline{V} \subset B_m$. Find a B_n inside of V, and we are now done: namely, the function $g_{n,m}$ is what we were after. ■

We now come back to a result that we previously put off.

Theorem 34.3. *Suppose X is locally compact, Hausdorff, and second-countable. Then X is normal.*

Proof. Given closed, disjoint subsets A and B, we want to separate them using disjoint open sets.

Consider first the case where $A = \{a\}$ is a point. Writing $V = X \setminus B$, we have $a \in V$, and we wish to find U with $a \in U \subseteq \overline{U} \subseteq V$. Since X is locally compact, Hausdorff, we can consider the one-point compactificaiton $\widehat{X}$. But now we have $a \in V \subseteq \widehat{X}$, and $\widehat{X}$ is compact Hausdorff and therefore normal. So we get the desired U. Note that the same argument does not work for a general A, since we would not know that A is closed in $\widehat{X}$ (unless A is compact). We have proved that X is regular (T_3).

Now let A and B be general closed, disjoint subsets. For each $a \in A$, we can find a basis element U_a with $a \in U_a \subseteq \overline{U_a} \subset X \setminus B$. Since our basis is countable, we can enumerate all such U_a's to get a countable cover $\{U_n\}$ of A which is disjoint from B. Similarly, we get a countable cover $\{V_n\}$ of B which is disjoint from A. But the U_n's need not be disjoint from the V_k's so we need to fix this.

Define new covers of A and B, respectively, as follows. For each n, define

$$\widehat{U}_n = U_n \setminus \bigcup_{j=1}^{n} \overline{V}_j \qquad \text{and} \qquad \widehat{V}_n = V_n \setminus \bigcup_{j=1}^{n} \overline{U}_j$$

The $\widehat{U}_n$'s still cover A because we have removed the $\overline{V}_j$, which were all disjoint from A. Similarly, the $\widehat{V}_n$ cover B. Moreover, $\widehat{U}_n$ is disjoint from $\widehat{V}_j$ because, assuming WLOG that $n < j$, the closure of U_n has been removed from V_j in the formation of $\overline{V}_j$. ■

Combining the previous results gives

Corollary 34.4. *Suppose X is locally compact, Hausdorff, and second-countable. Then X is metrizable.*

35. Monday, Nov. 17

Exam day.

36. Wednesday, Nov. 19

We finally arrive at one of the most important definitions of the course.

Definition 36.1. A (topological) n**-manifold** M is a Hausdorff, second-countable space such that each point has a neighborhood homeomorphic to an open subset of $\mathbb{R}^n$.

Example 36.2. (1) $\mathbb{R}^n$ and any open subset is obviously an n-manifold

(2) S^1 is a 1-manifold. More generally, S^n is an n-manifold. Indeed, we have shown that if you remove a point from S^n, the resulting space is homeomorphic to $\mathbb{R}^n$.

(3) T^n, the n-torus, is an n-manifold. In general, if M is an m-manifold and N is an n-manifold, then $M \times N$ is an $(m+n)$-manifold.

(4) $\mathbb{RP}^n$ is an n-manifold. There is a standard covering of $\mathbb{RP}^n$ by open sets as follows. Recall that $\mathbb{RP}^n = (\mathbb{R}^{n+1} \setminus \{0\})/\mathbb{R}^\times$. For each $1 \le i \le n+1$, let $V_i \subseteq \mathbb{R}^{n+1} \setminus \{0\}$ be the complement of the hyperplane $x_i = 0$. This is an open, saturated set, and so its image $U_i = V_i/\mathbb{R}^\times \subseteq \mathbb{RP}^n$ is open. The V_i's cover $\mathbb{R}^{n+1} \setminus \{0\}$, so the U_i's cover $\mathbb{RP}^n$. We leave the rest of the details as an exercise.

(5) $\mathbb{CP}^n$ is a $2n$-manifold. This is similar to the description given above.

(6) $O(n)$ is a $\frac{n(n-1)}{2}$-manifold. Since it is also a topological group, this makes it a *Lie group*. The standard way to see that this is a manifold is to realize the orthogonal group as the preimage of the identity matrix under the transformation $M_n(R) \longrightarrow M_n(R)$ that sends A to $A^T A$. This map lands in the subspace $S_n(R)$ of symmetric $n \times n$ matrices. This space can be identified with $\mathbb{R}^{n(n+1)/2}$.

Now the $n \times n$ identity matrix is an element of S_n, and an important result in differential topology (Sard's theorem) that says that if a certain derivative map is surjective, then the preimage of the submanifold $\{I_n\}$ will be a submanifold of $M_n(\mathbb{R})$ of the same "codimension". in this case, the relevant derivative is the matrix of partial derivatives of $A \mapsto A^T A$, writen in a suitable basis. It follows that

$$\dim O(n) = n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}.$$

The dimension statement can also be seen directly as follows. If A is an orthogonal matrix, its first column is just a point of S^{n-1}. Then its second column is a point on the sphere orthogonal to the first column, so it lives in an "equator", meaning a sphere of dimension one less. Continuing in this way, we see that the "degree of freedom" for specifying a point of $O(n)$ is $(n-1) + (n-2) + \cdots + 1 = \frac{n(n-1)}{2}$.

(7) $\mathrm{Gr}_{k,n}(\mathbb{R})$ is a $k(n-k)$-manifold. One way to see this is to use the homeomorphism

$$\mathrm{Gr}_{k,n}(\mathbb{R}) \cong O(n)/\big(O(k) \times O(n-k)\big)$$

from Example 18.1. We get

$$\begin{aligned}\dim \mathrm{Gr}_{n,k}(\mathbb{R}) &= \dim O(n) - \big(\dim O(k) + \dim O(n-k)\big) \\ &= \sum_{j=1}^{n-1} j - \left(\sum_{j=1}^{k-1} j + \sum_{\ell=1}^{n-k-1} \ell\right) = \sum_{j=k}^{n-1} j - \sum_{\ell=1}^{n-k-1} \ell \\ &= \sum_{\ell=0}^{n-k-1} k+\ell - \sum_{\ell=0}^{n-k-1} \ell = \sum_{\ell=0}^{n-k-1} k = k(n-k)\end{aligned}$$

Here are some nonexamples of manifolds.

Example 36.3. (1) The union of the coordinate axes in $\mathbb{R}^2$. Every point has a neighborhood like $\mathbb{R}^1$ except for the origin.
(2) A discrete uncountable set is not second countable.
(3) A 0-manifold is discrete, so $\mathbb{Q}$ is not a 0-manifold.
(4) Glue together two copies of $\mathbb{R}$ by identifying any nonzero x in one copy with the point x in the other. This is second-countable and looks locally like $\mathbb{R}^1$, but it is not Hausdorff.

37. Fri, Nov. 21

Proposition 37.1. *Any manifold is normal.*

Proof. This follows form Theorem 34.3. To see that a manifold M is locally compact, consider a point $x \in M$. Then x has a Euclidean neighborhood $x \in U \subseteq M$. U is homeomorphic to an open subset V of $\mathbb{R}^n$, so we can find a compact neighborhood K of x in V (think of a closed ball in $\mathbb{R}^n$). Under the homeomorphism, K corresponds to a compact neighborhood of x in U. ∎

It also follows similarly that any manifold is metrizable, but we can do better. It is convenient to introduce the following term.

Recall that the **support** of a continuous function $f : X \longrightarrow \mathbb{R}$ is $\mathrm{supp}(f) = \overline{f^{-1}(\mathbb{R} \setminus \{0\})}$.

Definition 37.2. Let $\mathcal{U} = \{U_1, \dots U_n\}$ be a finite cover of X. A **partition of unity** subordinate to $\mathcal{U}$ is a collection $\varphi_j \longrightarrow [0,1]$ of continuous functions such that

(1) $\mathrm{supp}(\varphi_\alpha) \subseteq U_\alpha$
(2) we have $\sum_j \varphi_j = 1$.

Theorem 37.3. *Let $\mathcal{U} = \{U_1, \dots, U_n\}$ be a finite covering of the normal space X. Then there is a partition of unity subordinate to $\mathcal{U}$.*

Lemma 37.4. *Let $\mathcal{U} = \{U_1, \dots, U_n\}$ be a finite covering of the normal space X. Then there is a finite cover $\mathcal{V} = \{V_1, \dots, V_n\}$ such that $V_i \subseteq \overline{V_i} \subseteq U_i$ for all i.*

Proof. We give only the argument in the case $n = 2$. Let $A = X \setminus U_2$. Then $A \subseteq U_1$, so we can find an open V_1 with $A \subseteq V_1 \subseteq \overline{V_1} \subseteq U_1$. Now $\{V_1, U_2\}$ is an open cover of X. In the same way, we replace U_2 be a V_2 with $X \setminus V_1 \subseteq V_2 \subseteq \overline{V_2} \subseteq U_2$. ∎

Proof of Theorem 37.3. We use the lemma twice, to get finite covers $\{V_1, \dots, V_n\}$ and $\{W_1, \dots, W_n\}$ with

$$W_i \subseteq \overline{W_i} \subseteq V_i \subseteq \overline{V_i} \subseteq U_i$$

for all i. For each i we have a Urysohn function $g_i : X \longrightarrow [0,1]$ with $g_i \equiv 1$ on $\overline{W_i}$ and vanishing outside of V_i. Note that this implies that $\mathrm{supp}(g_i) \subseteq \overline{V_i} \subseteq U_i$. Since the W_i cover X, it follows that if we define $G = \sum_i g_i$, then $G(x) \geq 1$ for all x. Thus $\varphi_i = g_i/G$ is a continuous function taking values in $[0, 1]$, and we get

$$\sum_i \varphi_i = \sum_i \frac{g_i}{G} = \frac{\sum_i g_i}{\sum_i g_i} = 1.$$

■

Theorem 37.5. *Any manifold M^n admits an embedding into some Euclidean space $\mathbb{R}^N$.*

Proof. The theorem is true as stated, but we only prove it in the case of a compact manifold. Note that in this case, since M is compact and $\mathbb{R}^N$ is Hausdorff, it is enough to find a continuous injection of M into some $\mathbb{R}^N$.

Since M is a manifold, it has an open cover by sets that are homeomorphic to $\mathbb{R}^n$. Since it is compact, there is a finite subcover $\{U_1, \dots, U_k\}$. By Theorem 37.3, there is a partition of unity $\{\varphi_1, \dots, \varphi_k\}$ subordinate to this cover. For each i, let $f_i : U_i \xrightarrow{\cong} \mathbb{R}^n$ be a homeomorphism. We can then piece these together as follows: for each $i = 1, \dots, k$, define $g_i : M \longrightarrow \mathbb{R}^n$ by

$$g_i(x) = \begin{cases} f_i(x)\varphi_i(x) & x \in U_i \\ \mathbf{0} & x \in X \setminus \operatorname{supp}(\varphi_i) \end{cases}.$$

Note that g_i is continuous by the glueing lemma, since $\operatorname{supp}(\varphi_i)$ is closed. Then the k functions g_i together give a continuous function $g : M \longrightarrow \mathbb{R}^{nk}$. Unfortunately, this need not be injective, since if $f_i(x) = \mathbf{0}$ and x does not lie in any other U_j, it follows that $g(x) = \mathbf{0}$. Since there can be more than one such x, we cannot conclude that g is injective.

One way to fix this would be to stick on the functions φ_i, in order to separate out points lying in different U_i's. Define $G = (g_1, \dots, g_k, \varphi_1, \dots, \varphi_k) : M \longrightarrow \mathbb{R}^{nk+k}$. But now G is injective, since if $G(x) = G(x')$ and we pick i so that $\varphi_i(x) = \varphi_i(x') > 0$, then this means that x and x' both lie in U_i. But then $g_i(x) = g_i(x')$ so $f_i(x) = f_i(x')$. Since f_i is a homeomorphism, it follows that $x = x'$. ■

In fact, one can do better. Munkres shows (Cor. 50.8) that every compact n-manifold embeds inside $\mathbb{R}^{2n+1}$.

Last time, we discussed some of the nice properties of manifolds. Here is one more we did not get to.

Proposition 38.1. *Any manifold is locally path-connected.*

This follows immediately since a manifold is locally Euclidean.

Another related concept is that of paracompactness. This is especially important in the theory of manifolds and vector bundles. We make a couple of preliminary definitions first.

Definition 38.2. If $\mathcal{U}$ and $\mathcal{W}$ are collections of subsets of X, we say that $\mathcal{W}$ is a **refinement** of $\mathcal{U}$ if every $W \in \mathcal{W}$ is a subset of some $U \in \mathcal{U}$.

Definition 38.3. An open cover $\mathcal{U}$ of X is said to be **locally finite** if every $x \in X$ has a neighborhood meeting only finitely many elements of the cover.

For example, the covering $\{(n, n+2) \mid n \in \mathbb{Z}\}$ of $\mathbb{R}$ is locally finite.

Definition 38.4. A space X is said to be **paracompact** if every open cover has a locally finite refinement.

From the definition, it is clear that compact implies paracompact. But this really is a generalization, as the next example shows.

Proposition 38.5. *The space $\mathbb{R}$ is paracompact.*

Proof. Let $\mathcal{U}$ be an open cover of $\mathbb{R}$. For each $n \geq 0$, let $A_n = \pm[n, n+1]$ and $W_n = \pm(n-\frac{1}{2}, n+\frac{3}{2})$. Then $A_n \subset W_n$, A_n is compact and W_n is open. (We take $W_0 = (-\frac{3}{2}, \frac{3}{2})$.) Fix an n. For each $x \in A_n$, pick a $U_x \in \mathcal{U}$ with $x \in U_x$, and let $V_x = U_x \cap W_n$. The V_x's give an open cover of A_n, and so there is a finite collection $\mathcal{V}_n$ of V_x's that will cover A_n. Then $\mathcal{V} = \bigcup_n \mathcal{V}_n$ gives a locally finite refinement of $\mathcal{U}$. (Note that only W_{n-1}, W_n, and W_{n+1} meet the subset A_n). ■

This argument adapts easily to show that $\mathbb{R}^n$ is paracompact. In fact, something more general is true.

Lemma 38.6. *Any open cover of a second countable space has a countable subcover.*

Proof. Given a countable basis $\mathcal{B}$ and an open cover $\mathcal{U}$, we first replace the basis by the countable subset $\mathcal{B}'$ consisting of those basis elements that are entirely contained in some open set from the cover (this is a basis too, but we don't need that). For each $B \in \mathcal{B}'$, pick some $U_B \in \mathcal{U}$ containing B, and let $\mathcal{U}' \subseteq \mathcal{U}$ be the (countable) collection of such U_B. It only remains to observe that $\mathcal{U}'$ is still a cover, because

$$\bigcup_{\mathcal{U}'} U_B \supset \bigcup_{\mathcal{B}'} B = X.$$

■

Proposition 38.7. *Every second countable, locally compact Hausdorff space is paracompact.*

The proof strategy is the same. The assumptions give you a cover (basis) by precompact sets and thus a countable cover by precompact sets. You use this to manufacture a countable collection of compact sets A_n and open sets W_n that cover X as above. The rest of the proof is the same.

Note that of the assumptions in the proposition, locally compact and Hausdorff are both *local* properties, whereas second countable is a global property. As we will see, paracompactness (and therefore the assumptions in this proposition) is enough to guarantee the existence of some nice functions on a space.

Corollary 38.8. *Any manifold is paracompact.*

Theorem 38.9 (Munkres, Theorem 41.4)**.** *If X is metric, then it is paracompact.*

Next, we show that paracompact and Hausdorff implies normal. First, we need a lemma.

Lemma 38.10. *If $\{A\}$ is a locally finite collection of subsets of X, then*

$$\overline{\bigcup A} = \bigcup \overline{A}.$$

Proof. We have already shown before that the inclusion $\supset$ holds generally. The other implication follows from the neighborhood criterion for the closure. Let $x \in \overline{\bigcup A}$. Then we can find a neighborhood U of x meeting only $A_1, \ldots, A_n$. Then $x \in \overline{\bigcup_{i=1}^n A_i}$ since else there would be a neighborhood V of x missing the A_i's. Then $U \cap V$ would be a neighborhood missing $\bigcup A$. But $\overline{\bigcup_{i=1}^n A_i} = \bigcup_{i=1}^n \overline{A_i}$, so we are done. ■

Theorem 38.11 (Lee, Theorem 4.81)**.** *If X is paracompact and Hausdorff, then it is normal.*

Proof. We first use the Hausdorff assumption to show that X is regular. A similar argument can then be made, using regularity, to show normality.

Thus let A be closed and $b \notin A$. We wish to find disjoint open sets $A \subseteq U$ and $b \in V$. For every $a \in A$, we can find disjoint open neighborhoods U_a of a and V_a of b. Then $\{U_a\} \cup \{X \setminus A\}$ is an open cover, so there is a locally finite subcover $\mathcal{V}$. Take $\mathcal{W} \subseteq \mathcal{V}$ to be the $W \in \mathcal{V}$ such that $W \subseteq U_a$ for some a. Then $\mathcal{W}$ is still locally finite.

We take $U = \bigcup_{W \in \mathcal{W}} W$ and $V = X \setminus \overline{U}$. We know $b \in V$ since $\overline{U} = \bigcup \overline{W}$, and $b \notin \overline{W}$ since $W \subseteq U_a$ and b has a neighborhood (V_a) disjoint from U_a. ■

Definition 38.12. Let $\mathcal{U} = \{U_\alpha\}$ be a cover of X. A **partition of unity** subordinate to $\mathcal{U}$ is a collection $\varphi_\alpha : X \longrightarrow [0,1]$ of continuous functions such that

1. $\operatorname{supp}(\varphi_\alpha) \subseteq U_\alpha$
2. the collection $\operatorname{supp}(\varphi_\alpha)$ is locally finite
3. we have $\sum_\alpha \varphi_\alpha = 1$. Note that, when evaluated at some $x \in X$, this sum is always finite by the local finite assumption (2).

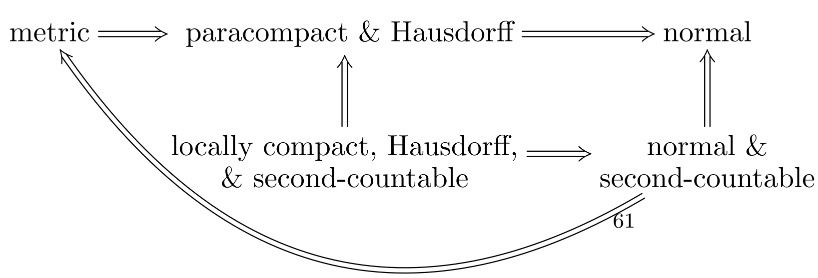
Theorem 38.13. *Let X be paracompact Hausdorff, and let $\mathcal{U} = \{U_\alpha\}$ be an open cover. Then there exists a partition of unity subordinate to $\mathcal{U}$.*

Lemma 38.14 (Lee, 4.84)**.** *There exists a locally finite refinement $\{V_\alpha\}$ of $\{U_\alpha\}$ with $\overline{V_\alpha} \subseteq U_\alpha$.*

Proof of Theorem. We apply the lemma twice to get locally finite covers $\{V_\alpha\}$ and $\{W_\alpha\}$ with $\overline{W_\alpha} \subseteq V_\alpha \subseteq \overline{V_\alpha} \subseteq U_\alpha$. For each α, we use Urysohn's lemma to get $f_\alpha : X \longrightarrow [0,1]$ with $f_\alpha \equiv 1$ on $\overline{W_\alpha}$ and $\operatorname{supp}(f_\alpha) \subseteq \overline{V_\alpha} \subseteq U_\alpha$. Since $\{V_\alpha\}$ is locally finite, we can define $f : X \longrightarrow [0,1]$ by $f = \sum_\alpha f_\alpha$. Locally around some $x \in X$, the function f is a finite sum of f_α's, and so is continuous. It only remains to normalize our f_α's. Note that at any $x \in X$, we can find an α for which $x \in W_\alpha$, and so $f(x) \geq f_\alpha(x) = 1$. Thus it makes sense to define $\varphi_\alpha : X \longrightarrow [0,1]$ by

$$\varphi_\alpha(x) = \frac{f_\alpha(x)}{f(x)}.$$

We have $\operatorname{supp}(\varphi_\alpha) = \operatorname{supp}(f_\alpha)$, and so the φ_α give a partition of unity. ■



39. Mon, Dec. 1

The last main topic from the introductory part of the course on metric spaces is the idea of a function space. Given any two spaces A and Y, we will want to be able to define a topology on the set of continuous functions $A \longrightarrow Y$ in a sensible way. We already know one topology on Y^A, namely the product topology. But this does not use the topology on A at all.

Let's forget about topology for a second. Recall from the beginning of the course that a function $h : X \times A \longrightarrow Y$ between sets is equivalent to a function

$$\Psi(h) : X \longrightarrow Y^A.$$

Given h, the map $\Psi(h)$ is defined by $\big(\Psi(h)(x)\big)(a) = h(x, a)$. Conversely, given $\Psi(h)$, the function h can be recovered by the same formula.

Let's play the same game in topology. What we want to say is that a continuous map $h : X \times A \longrightarrow Y$ is the same as a continuous map $X \longrightarrow \mathrm{Map}(A, Y)$, for some appropriate *space* of maps $\mathrm{Map}(A, Y)$. Let's start by seeing why the product topology does *not* have this property.

We write $\mathcal{C}(X, Z)$ for the *set* of continuous maps $X \longrightarrow Z$. It is not difficult to check that the set-theoretic construction from above does give a function

$$\mathcal{C}(X \times A, Y) \longrightarrow \mathcal{C}(X, Y^A),$$

where for the moment Y^A denotes the set of continuous functions $A \longrightarrow Y$ given the product topology. But this function is not surjective.

Example 39.1. Take $A = [0, 1]$, $Y = \mathbb{R}$, and $X = Y^A = \mathbb{R}^{[0,1]}$. We can consider the identity map $\mathbb{R}^{[0,1]} \longrightarrow \mathbb{R}^{[0,1]}$. We would like this to correspond to a continuous map $\mathbb{R}^{[0,1]} \times [0, 1] \longrightarrow \mathbb{R}$. We see that, ignoring the topology, this function must be the evaluation function $ev : (g, x) \mapsto g(x)$. But this is not continuous.

To see this consider $ev^{-1}((0, 1))$. The point $(\mathrm{id}, 1/2)$ lies in this preimage, but we claim that no neighborhood of this point is contained in the preimage. In fact, we claim no basic neighborhood $U \times (a, b)$ lies in the preimage. For such a U must consist of functions that are close to $\mathrm{id} : [0, 1] \longrightarrow \mathbb{R}$ at *finitely many points* $c_1, \ldots, c_n$. So given any such U and any interval $(a, b) = (1/2 - \epsilon, 1/2 + \epsilon)$, pick any point $d \in (a, b)$ that is distinct from the c_i. It is simple to construct a continuous function $g : [0, 1] \longrightarrow \mathbb{R}$ such that (1) $g(c_i) = c_i$ for each i and (2) $g(d) = 2$. Then $(g, d) \in U \times (a, b)$ but $(g, d) \notin ev^{-1}((0, 1))$ since $ev(g, d) = g(d) = 2$.

The **compact-open** topology on the set $\mathcal{C}(A, Y)$ has a prebasis given by

$$S(K, U) = \{f : A \longrightarrow Y \mid f(K) \subseteq U\},$$

where K is compact and $U \subseteq Y$ is open. We write $\mathrm{Map}(A, Y)$ for the set $\mathcal{C}(A, Y)$ equipped with the compact-open topology.

Theorem 39.2. *Suppose that A is locally compact Hausdorff. Then a function $f : X \times A \longrightarrow Y$ is continuous if and only if the induced function $g = \Psi(f) : X \longrightarrow \mathrm{Map}(A, Y)$ is continuous.*

Proof. ($\Rightarrow$) This direction does not need that A is locally compact. Before we give the proof, we should note why $\Psi(f)(x) : A \longrightarrow Y$ is continuous. This map is the composite $A \xrightarrow{\iota_x} X \times A \xrightarrow{f} Y$ and therefore continuous.

We now wish to show that $g = \Psi(f)$ is continuous. Let $S(K, U)$ be a sub-basis element in $\mathrm{Map}(A, Y)$. We wish to show that $g^{-1}(S(K, U))$ is open in X. Let $g(x) = f(x, -) \in S(K, U)$. Since f is continuous, the preimage $f^{-1}(U) \subseteq X \times A$ is open. Furthermore, $\{x\} \times K \subseteq f^{-1}(U)$. We wish to use the Tube Lemma, so we restrict from $X \times A$ to $X \times K$. By the Tube Lemma, we can find a basic neighborhood V of x such that $V \times K \subseteq (X \times K) \cap f^{-1}(U)$. It follows that $g(V) \subseteq S(K, U)$, so that V is a neighborhood of x in $g^{-1}(S(K, U))$.

(⇐) Suppose that g is continuous. Note that we can write f as the composition

$$X \times A \xrightarrow{g\times \mathrm{id}} \mathrm{Map}(A,Y) \times A \xrightarrow{ev} Y,$$

so it is enough to show that ev is continuous.

Lemma 39.3. *The map $ev : \mathrm{Map}(A,Y)\times A \longrightarrow Y$ is continuous if A is locally compact Hausdorff.*

Proof. Let $U \subseteq Y$ be open and take a point (f,a) in $ev^{-1}(U)$. This means that $f(a) \in U$. Since A is locally compact Hausdorff, we can find a compact neighborhood K of a contained in $f^{-1}(U)$ (this is open since f is continuous). It follows that $S(K,U)$ is a neighborhood of f in $\mathrm{Map}(A,Y)$, so that $S(K,U) \times K$ is a neighborhood of (f,a) in $ev^{-1}(U)$. ■

■

40. Wed, Dec. 3

Even better, we have

Theorem 40.1. *Let X and A be locally compact Hausdorff. Then the above maps give homeomorphisms*

$$\mathrm{Map}(X \times A, Y) \cong \mathrm{Map}(X, \mathrm{Map}(A,Y)).$$

It is fairly simple to construct a continuous map in either direction, using Theorem 39.2. You should convince yourself that the two maps produced are in fact inverse to each other.

In practice, it's a bit annoying to keep track of these extra hypotheses at all times, especially since not all constructions will preserve these properties. It turns out that there is a "convenient" category of spaces, where everything works nicely.

Definition 40.2. A space A is **compactly generated** if a subset $B \subseteq A$ is closed if and only if for every map $u : K \longrightarrow A$, where K is compact Hausdorff, then $u^{-1}(B) \subseteq K$ is closed.

We say that the topology of A is determined (or generated) by compact subsets. Examples of compactly generated spaces include locally compact spaces and first countable spaces.

Definition 40.3. A space X is **weak Hausdorff** if the image of every $u : K \longrightarrow X$ is closed in X.

There is a way to turn any space into a weak Hausdorff compactly generated space. In that land, <u>everything</u> works well! For the most part, whenever an algebraic topologist says "space", they really mean a compactly generated weak Hausdorff space. Next semester, we will always implicity be working with spaces that are CGWH.

Looking back to the initial discussion of metric spaces, there we introduced the uniform topology on a mapping space.

Theorem 40.4 (Munkres, 46.7 or Willard, 43.6)**.** *Let Y be a metric space. Then on the set $\mathcal{C}(A,Y)$ of continuous functions $A \longrightarrow Y$, the compact-open topology is intermediate between the uniform topology and the product topology. Furthermore, the compact-open topology agrees with the uniform topology if A is compact.*

The main point is to show (Munkres, Theorem 46.8) that the compact-open topology can be described by basis elements

$$B_K(f,\epsilon) = \{g : A \longrightarrow Y \mid \sup_K d(f(x), g(x)) < \epsilon\}.$$

To see that this satisfies the intersection property for a basis, suppose that

$$g \in B_{K_1}(f_1,\epsilon_1) \cap B_{K_2}(f_2,\epsilon_2)$$

Write $m_i = \sup_{K_i} d(f_i(x), g(x))$ and $\delta_i = \epsilon_i - m_i$. Then the triangle inequality gives

$$B_{K_i}(g, \delta_i) \subseteq B_{K_i}(f_i, \epsilon_i).$$

It follows that, setting $\delta = \min\{\delta_1, \delta_2\}$

$$g \in B_{K_1 \cup K_2}(g, \delta) \subseteq B_{K_1}(f_1, \epsilon_1) \cap B_{K_2}(f_2, \epsilon_2).$$

In the setting of metric spaces, the compact-open topology is known as the topology of compact convergence, as convergence of functions corresponds to (uniform) convergence on compact subsets.

One of the good properties of the uniform topology is that tif (f_n) is sequence of continuous functions and $f_n \to f$ in the uniform topology, then f is continuous. In other words, if we denote by $\mathcal{F}(X, Y)$ the set of *all* functions $X \longrightarrow Y$, then

$$\mathcal{C}(X, Y)_{\text{unif}} \subseteq \mathcal{F}(X.Y)_{\text{unif}}$$

is closed. This also happens in the compact-open topology.

Proposition 40.5. *Suppose that X is locally compact. Then*

$$\text{Map}(X, Y) \subseteq \mathcal{F}(X, Y)_{\text{compact}-\text{open}}$$

is closed.

For fun, here is one of the first results towards the theory of C^*-algebras (pronounced C-star).

Theorem 40.6. *Let X be compact Hausdorff and denote by $C(X)$ the space* $\text{Map}(X, \mathbb{R})$ *of real-valued functions on X. Then the map*

$$\Lambda : X \longrightarrow \widehat{C(X)} = \{\lambda : C(X) \longrightarrow \mathbb{R} \mid \lambda \textit{ is a continuous } \mathbb{R}\textit{-algebra map}\}$$

defined by

$$\Lambda(x) = \text{ev}_x$$

is a homeomorphism if $\widehat{C(X)} \subseteq \prod_{C(X)} \mathbb{R}$ is equipped with the product topology.

In this case, the product topology coincides with a topology of interest in analysis known as the weak-$*$ topology.

Proof. Since we have given $\widehat{C(X)}$ the product topology, it is simple to verify that Λ is continuous. Note that since X is compact and $\widehat{C(X)}$ is Hausdorff, it remains only to show that Λ is a bijection.

Suppose that $\Lambda(x) = \Lambda(x')$. Since X is compact Hausdorff, for any two distinct points there is a continuous function taking different values at those points. The fact that $\Lambda(x) = \Lambda(x')$ says that no such function exists for x and x', so we must have $x = x'$.

Now let $\lambda \in \widehat{C(X)}$. We wish to show that $\lambda = \text{ev}_x$ for some x.

The main step is to show that there exists $x \in X$ such that if $\lambda(f) = 0$ for some f, then $f(x) = 0$. Suppose not. Then for every $x \in X$, there exists a function f_x with $\lambda(f_x) = 0$ but $f_x(x) \neq 0$. For each x, let $U_x = f_x^{-1}(\mathbb{R} \setminus \{0\})$. Then the collection $\{U_x\}$ covers X since $x \in U_x$. As X is compact, there is a finite subcover $\{U_{x_1}, \ldots, U_{x_n}\}$. Now define

$$g = f_{x_1}^2 + \cdots + f_{x_n}^2,$$

and note that $g(x) > 0$ for all x. This is because $f_{x_i} \neq 0$ on U_{x_i} and the U_{x_i} cover X. Since g is nonzero, it follows that $1/g$ is also continuous on X. But now

$$1 = \lambda(g \cdot 1/g) = \lambda(g) \cdot \lambda(1/g),$$

which implies that $\lambda(g) \neq 0$. But λ is an algebra homomorphism, so

$$\lambda(g) = \sum_i \lambda(f_{x_i})^2 > 0,$$

which is a contradiction.

This now establishes that there must be an $x \in X$ such that if $\lambda(f) = 0$ then $f(x) = 0$. But now the theorem follows, for if $f \in C(X)$, then

$$\lambda(f - \lambda(f) \cdot 1) = \lambda(f) - \lambda(f) \cdot \lambda(1) = 0.$$

By the above, we then have that $f(x) - \lambda(f) = 0$, so that $\lambda(f) = f(x)$. In other words, $\lambda = \mathrm{ev}_x$. ∎

Recently, we consider topological manifolds, which are a nice collection of spaces. Next semester, you will often work with another nice collection of spaces that can be built inductively. These are cell complexes, or CW complexes.

A typical example is a sphere. In dimension 1, we have S^1, which we can represent as the quotient of $I = [0, 1]$ by endpoint identification. Another way to say this is that we start with a point, and we "attach" an interval to that point by gluing both ends to the given point.

For S^2, there are several possibilities. One is to start with a point and glue a disk to the point (glueing the boundary to the point). An alternative is to start with a point, then attach an interval to get a circle. To this circle, we can attach a disk, but this just gives us a disk again, which we think of as a hemisphere. If we then attach a second disk (the other hemisphere), we get S^2.

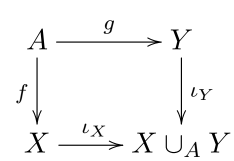
But what do we really mean by "attach a disk"?

Let's start today by discussing the general "pushout" construction.

Definition 40.7. Suppose that $f : A \longrightarrow X$ and $g : A \longrightarrow Y$ are continuous maps. The **pushout** (or glueing construction) of X and Y along A is defined as

$$X \cup_A Y := X \amalg Y / \sim, \qquad f(a) \sim g(a).$$

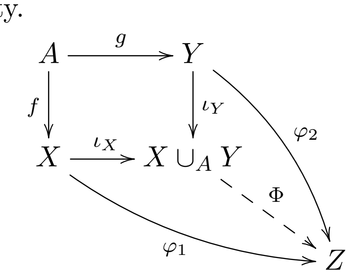
We have an inclusion $X \hookrightarrow X \amalg Y$. Composing this with the quotient map to $X \cup_A Y$ gives the map $\iota_X : X \longrightarrow X \cup_A Y$. We similarly have a map $\iota_Y : Y \longrightarrow X \cup_A Y$. Moreover, these maps make the diagram to the right commute. The point is that



$$\iota_X(f(a)) = \overline{f(a)} = \overline{g(a)} = \iota_Y(g(a)).$$

The main point of this construction is the following property.

Proposition 40.8 (Universal property of the pushout)**.** *Suppose that $\varphi_1 : X \longrightarrow Z$ and $\varphi_2 : Y \longrightarrow Z$ are maps such that $\varphi_1 \circ f = \varphi_2 \circ g$. Then there is a unique map $\Phi : X \cup_A Y \longrightarrow Z$ which makes the diagram to the right commute.*



This generalizes the "pasting" lemma. Suppose that $U, V \subseteq X$ are open subsets. Then it is not difficult to show that the pushout $U \cup_{U \cap V} V$ is homeomorphic to X. The universal property for the pushout then says that specifying a continuous map out of X is the same as specifying a pair of continuous maps out of U and V which agree on their intersection $U \cap V$. This is precisely the statement of the pasting lemma!

Definition 40.9. (Attaching an interval) Given a space X and two points $x \neq y \in X$, we get a continuous map $\alpha : S^0 \longrightarrow X$ with $\alpha(0) = x$ and $\alpha(1) = y$. There is the standard inclusion $S^0 \hookrightarrow D^1 = [-1, 1]$, and we write $X \cup_\alpha D^1$ for the pushout

$$\begin{array}{ccc} S^0 & \longrightarrow & D^1 \\ {\scriptstyle \alpha}\downarrow & & \downarrow \\ X & \xrightarrow{\iota_X} & X \cup_\alpha D^1 \end{array}$$

The image $\iota(\mathrm{Int}(D^1))$ is referred to as a 1-cell and is sometimes denoted e^1. Thus the above space, which is described as obtained by attaching an 1-cell to X, is also written $X \cup_\alpha \overline{e^1}$ or $X \cup_\alpha e^1$.

41. Fri, Dec. 5

Generalizing the construction from last time, for any n, we have the standard inclusion $S^{n-1} \hookrightarrow D^n$ as the boundary.

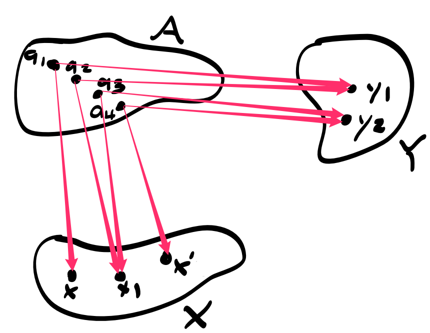
Definition 41.1. Given a space X and a continuous map $\alpha : S^{n-1} \longrightarrow X$, we write $X \cup_\alpha D^n$ for the pushout

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & D^n \\ {\scriptstyle \alpha}\downarrow & & \downarrow \\ X & \xrightarrow{\iota_X} & X \cup_\alpha D^n \end{array}$$

The image $\iota(\mathrm{Int}(D^n))$ is referred to as an n-cell and is sometimes denoted e^n. Thus the above space, which is described as obtained by attaching an n-cell to X, is also written $X \cup_\alpha \overline{e^n}$ or $X \cup_\alpha e^n$.

In general, this attaching process does not disturb the interiors of the cells, as follows from

Proposition 41.2. *If $g : A \hookrightarrow Y$ is injective, then $\iota_X : X \longrightarrow X \cup_A Y$ is also injective.*



Proof. Suppose that $\iota_X(x) = \iota_X(x')$. The relation imposed on $X \amalg Y$ only affects points in $f(A)$ and $g(A)$. We assume that $x, x' \in f(A)$ since otherwise we must have $x = x'$. In general, the situation we should expect is represented in the picture to the right. But since g is injective, this means that $a_1 = a_2$ and $a_3 = a_4$. This implies that $x = f(a_1) = f(a_2) = x_1$ and that $x_1 = f(a_3) = f(a_4) = x'$. Putting these together gives $x = x'$. ■

Example 41.3. If $A = \emptyset$, then $X \cup_A Y = X \amalg Y$.

Example 41.4. If $A = *$, then $X \cup_A Y = X \vee Y$.

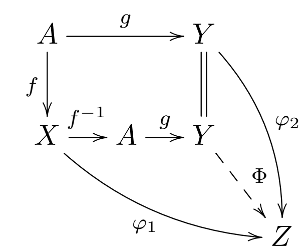
Example 41.5. If $A \subseteq X$ is a subspace and $Y = *$, then $X \cup_A * \cong X/A$.

By the way, Proposition 41.2 is not only true for injections.

Proposition 41.6. *(i) If $f : A \longrightarrow X$ is surjective, then so is $\iota_Y : Y \longrightarrow X \cup_A Y$.*
(ii) If $f : A \longrightarrow X$ is a homeomorphism, then so is $\iota_Y : Y \longrightarrow X \cup_A Y$.

Proof. We prove only (ii). We show that if f is a homeomorphism, then Y satisfies the same universal property as the pushout. Consider the test diagram to the right. We have no choice but to set $\Phi = \varphi_2$. Does this make the diagram commute? We need to check that $\Phi \circ g \circ f^{-1} = \varphi_1$. Well,

$$\Phi \circ g \circ f^{-1} = \varphi_2 \circ g \circ f^{-1} = \varphi_1 \circ f \circ f^{-1} = \varphi_1.$$



■

We use the idea of attaching cells (using a pushout) to inductively build up the idea of a cell complex or CW complex.

Definition 41.7. A **CW complex** is a space built in the following way

(1) Start with a discrete set X^0 (called the set of 0-cells, or the 0-skeleton)
(2) Given the $(n-1)$-skeleton X^{n-1}, the n-skeleton X^n is obtained by attaching n-cells to X^{n-1}.
(3) The space X is the union of the X^n, topologized using the "weak topology". This means that $U \subseteq X$ is open if and only if $U \cap X^n$ is open for all n.

The third condition is not needed if $X = X^n$ for some n (so that X has no cells in higher dimensions). On the other hand, the 'W' in the name CW complex refers to item 3 ("weak topology"). The 'C' in CW complex refers to the **C**losure finite property: the closure of any cell is contained in a finite union of cells. We will come back to this point later.

According to condition (2), the n-skeleton is obtained from the $(n-1)$-skeleton by attaching cells. Often, we think of this as attaching one cell at a time, but we can equally well attach them all at once, yielding a pushout diagram

$$\begin{array}{ccc} \coprod_{\mathcal{E}_n} S^{n-1} & \longrightarrow & \coprod_{\mathcal{E}_n} D^n \\ \downarrow & & \downarrow \\ X^{n-1} & \longrightarrow & X^n \end{array}$$

for each n. The maps $S^{n-1} \longrightarrow X^{n-1}$ are called the **attaching maps** for the cells, and the resulting maps $D^n \longrightarrow X^n$ are called the **characteristic maps**.

Example 41.8. (1) S^n. We have already discussed two CW structures on S^2. The first has X^0 a singleton and a single n-cell attached. The other had a single 0-cell and single 1-cell but two 2-cells attached. There is a third option, which is to start with two 0-cells, attach two 1-cells to get a circle, and then attach two 2-cells to get S^2.

The first and third CW structures generalize to any S^n. There is a minimal CW structure having a single 0-cell and single n-cell, and there is another CW structure have two cells in every dimension up to n.

42. Mon, Dec. 8

Last time, we were discussing CW complexes, and we considered two different CW structures on S^n. We continue with more examples.

(2) $\mathbb{RP}^n$. Let's start with $\mathbb{RP}^2$. Recall that one model for this space was as the quotient of D^2, where we imposed the relation $x \sim -x$ on the boundary. If we restrict our attention to the boundary S^1, then the resulting quotient is $\mathbb{RP}^1$, which is again a circle. The quotient map $q : S^1 \longrightarrow S^1$ is the map that winds twice around the circle. In complex coordinates, this would be $z \mapsto z^2$. The above says that we can represent $\mathbb{RP}^2$ as the pushout

$$\begin{array}{ccc} S^1 & \xrightarrow{\iota} & D^2 \\ {\scriptstyle q}\big\downarrow & & \big\downarrow \\ S^1 & \longrightarrow & \mathbb{RP}^2 \end{array}$$

If we build the 1-skeleton S^1 using a single 0-cell and a single 1-cell, then $\mathbb{RP}^2$ has a single cell in dimensions ≤ 2.

More generally, we can define $\mathbb{RP}^n$ as a quotient of D^n by the relation $x \sim -x$ on the boundary S^{n-1}. This quotient space of the boundary was our original definition of $\mathbb{RP}^{n-1}$. It follows that we can describe $\mathbb{RP}^n$ as the pushout

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{\iota} & D^n \\ {\scriptstyle q}\big\downarrow & & \big\downarrow \\ \mathbb{RP}^{n-1} & \longrightarrow & \mathbb{RP}^n \end{array}$$

Thus $\mathbb{RP}^n$ can be built as a CW complex with a single cell in each dimension $\leq n$.

(3) $\mathbb{CP}^n$. Recall that $\mathbb{CP}^1 \cong S^2$. We can think of this as having a single 0-cell and a single 2-cell. We defined $\mathbb{CP}^2$ as the quotient of S^3 by an action of S^1 (thought of as $U(1)$). Let $\eta : S^3 \longrightarrow \mathbb{CP}^1$ be the quotient map. What space do we get by attaching a 4-cell to $\mathbb{CP}^1$ by the map η? Well, the map η is a quotient, so the pushout $\mathbb{CP}^1 \cup_\eta D^4$ is a quotient of D^4 by the S^1-action on the boundary.

Now include D^4 into $S^5 \subseteq \mathbb{C}^3$ via the map

$$\varphi(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, x_4, \sqrt{1 - \sum x_i^2}, 0).$$

(This would be a hemi-equator.) We have the diagonal $U(1)$ action on S^5. But since any nonzero complex number can be rotated onto the positive x-axis, the image of φ meets every S^1-orbit in S^5, and this inclusion induces a homeomorphism on orbit spaces

$$D^4/U(1) \cong S^5/U(1) = \mathbb{CP}^2.$$

We have shown that $\mathbb{CP}^2$ has a cell structure with a single 0-cell, 2-cell, and 4-cell.

This story of course generalizes to show that any $\mathbb{CP}^n$ can be built as a CW complex having a cell in each even dimension.

43. Wednesday, Dec. 10

(4) (Torus) In general, a product of two CW complexes becomes a CW complex. We will describe this in the case $S^1 \times S^1$, where S^1 is built using a single 0-cell and single 1-cell.

Start with a single 0-cell, and attach two 1-cells. This gives $S^1 \vee S^1$. Now attach a single 2-cell to the 1-skeleton via the attaching map ψ defined as follows. Let us refer to the two circles in $S^1 \vee S^1$ as ℓ and r. We then specify $\psi : S^1 \longrightarrow S^1 \vee S^1$ by $\ell r \ell^{-1} r^{-1}$. What we mean is to trace out ℓ on the first quarter of the domain, to trace out r on the second quarter, to run ℓ in reverse on the third quarter, and finally to run r in reverse on the final quarter.

We claim that the resulting CW complex X is the torus. Since the attaching map $\psi : S^1 \longrightarrow S^1 \vee S^1$ is surjective, so is $\iota_{D^2} : D^2 \longrightarrow X$. Even better, it is a quotient map. On the other hand, we also have a quotient map $I^2 \longrightarrow T^2$, and using the homeomorphism $I^2 \cong D^2$ from before, we can see that the quotient relation in the two cases agrees. We say that this homeomorphism $T^2 \cong X$ puts a cell structure on the torus. There is a single 0-cell (a vertex), two 1-cells (the two circles in $S^1 \vee S^1$), and a single 2-cell.

Let's talk about some of the (nice!) topological properties of CW complexes.

Lemma 43.1. *Let $\mathcal{E} = \{e_i^{n_i}\}$ be the set of all cells in X. Then X is a quotient of $\coprod_{\mathcal{E}} D^{n_i}$. In particular, $A \subseteq X$ is open (or closed) if and only if, for each cell i and corresponding characteristic map $\Phi_i : D^{n_i} \longrightarrow X$, the preimage $\Phi_i^{-1}(A)$ is open (or closed) in D^{n_i}.*

Proof. The forward implication is clear by continuity of the φ_i. For the other direction, suppose that each $\Phi_i^{-1}(A)$ is open. Then $A \cap X^0$ is open in X^0, since X^0 is just the disjoint union of its cells. Now assume by induction that $A \cap X^{n-1}$ is open in X^{n-1}. But, by the construction of the pushout, the n-skeleton X^n is a quotient of $X^{n-1} \amalg \coprod D^n$. Since $A \cap X^n$ pulls back to an open set in each piece of this coproduct, it must be open in $A \cap X^n$ by the definition of the quotient topology. Now, since $A \cap X^n$ is open in X^n for all n, A is open in X by property W. ∎

Theorem 43.2. *Any CW complex X is normal.*

Proof. First, X is T_1 by the Lemma since any point obviously pulls back to a closed subset of every D_i^n. Let A and B be disjoint closed sets in X. We will show that X is normal by building a Urysohn function $f : X \longrightarrow [0,1]$ with $f(A) \equiv 0$ and $f(B) \equiv 1$. Because X satisfies property W, a function f defined on X is continuous if and only if its restriction to each X^n is continuous. We thus build the function f by building its restrictions f^n to X^n.

On X^0, we define

$$f^0(x) = \begin{cases} 0 & x \in A \cap X^0 \\ 1 & x \in B \cap X^0 \\ 1/2 & \text{else.} \end{cases}$$

Since X^0 is discrete, this is automatically continuous.

Now assume by induction that we have $f^{n-1} : X^{n-1} \longrightarrow [0,1]$ continuous with $f^{n-1}(A \cap X^{n-1}) \equiv 0$ and $f^{n-1}(B \cap X^{n-1}) \equiv 1$. Since we have a pushout diagram

$$\begin{array}{ccc} \coprod S^{n-1} & \longrightarrow & \coprod D^n \\ \downarrow & & \downarrow \\ X^{n-1} & \longrightarrow & X^n, \end{array}$$

by the universal property of the pushout, to define f^n on X^n, we need only specify a compatible pair of functions on X^{n-1} and on the disjoint union. On X^{n-1}, we take f^{n-1}. To define a map out of $\coprod D^n$, it is enough to define a map on each D^n.

For each n-cell e^i, define $W_i \subseteq D^n$ closed by $W_i = \partial D^n \cup \Phi_i^{-1}(A \cap X^n) \cup \Phi_i^{-1}(B \cap X^n)$. Define $g : W_i \longrightarrow [0,1]$ by

$$g(x) = \begin{cases} f^{n-1}(\varphi(x)) & x \in \partial D^n \\ 0 & x \in \Phi_i^{-1}(A \cap X^n) \\ 1 & x \in \Phi_i^{-1}(B \cap X^n). \end{cases}$$

We know that D^n is compact Hausdorff (or metric) and thus normal. Thus, by the Tietze extneion theorem (32.1) there is a Urysohn function for the disjoint closed sets $\Phi_i^{-1}(A \cap X^n)$ and $\Phi_i^{-1}(B \cap X^n)$ whose restriction to ∂D^n agrees with $f^{n-1} \circ \varphi_i$. Putting all of this together gives a Urysohn function on X^n for the $A \cap X^n$ and $B \cap X^n$. By induction, we are done. ■

44. Fri, Dec. 12

Even better,

Theorem 44.1 (Lee, Theorem 5.22)**.** *Every CW complex is paracompact.*

Proposition 44.2. *Any CW complex X is locally path-connected.*

Proof. Let $x \in X$ and let U be any open neighborhood of x. We want to find a path-connected neighborhood V of x in U. Recall that a subset $V \subseteq X$ is open if and only if $V \cap X^n$ is open for all n. We will define V by specifying open subsets $V^n \subseteq X^n$ with $V^{n+1} \cap X^n = V^n$ and then setting $V = \cup V^n$.

Suppose that x is contained in the cell e_i^n. We set $V^k = \emptyset$ for $k < n$. We specify V_n by defining $\Phi_j^{-1}(V^n)$ for each n-cell e_j^n. If $j \neq i$, we set $\Phi_j^{-1}(V_n) = \emptyset$. We define $\Phi_i^{-1}(V_n)$ to be an open n-disc around $\Phi_i^{-1}(x)$ whose closure is contained in $\Phi_i^{-1}(U)$. Now suppose we have defined V^k for some $k \geq n$. Again, we define V^{k+1} by defining each $\Phi_i^{-1}(V^{k+1})$. By assumption, $\overline{\Phi_i^{-1}(V^k)} \subseteq \partial D^{k+1} \subseteq \Phi_i^{-1}(U)$. By the Tube lemma, there is an $\epsilon > 0$ such that (using radial coordinates) $\Phi_i^{-1}(V^k) \times (1-\epsilon, 1] \subset U$. We define

$$\Phi_i^{-1}(V^{k+1}) = \Phi_i^{-1}(V^k) \times [1, 1-\epsilon/2),$$

which is path-connected by induction. This also guarantees that $\overline{V^{k+1}} \subset U \cap X^{k+1}$, allowing the induction to proceed. ■

Proposition 44.3 (Hatcher, A.1)**.** *Any compact subset K of a CW complex X meets finitely many cells.*

Proof. For each cell e_i meeting K, pick a point $k_i \in K \cap e_i$. Let $S = \{k_i\}$. We use proprety W to show that S is closed in X. It is clear that $S \cap X^0$ is closed in X^0 since X^0 is discrete. Assume that $S \cap X^{n-1}$ is closed in X^{n-1}. Now in X^n, the set $S \cap X^n$ is the union of the closed subset $S \cap X^{n-1}$ and the points k_i that lie in open n-cells. By Lemma 43.1, this set of k_i is closed as well.

The argument above in fact shows that any subset of S is closed, so that S is discrete. But S is closed in K, so S is compact. Since S is both discrete and compact, it must be finite. ■

Corollary 44.4. *Any CW complex has the closure-finite property, meaning that the closure of any cell meets finitely many cells.*

Proof. The closure of e_i is $\Phi_i(D_i^{n_i})$, which is compact. The result follows from the proposition. ■

Corollary 44.5.

(i) A CW complex X is compact if and only if it has finitely many cells.
(ii) A CW complex X is locally compact if and only if the collection $\mathcal{E}$ of cells is locally finite.

We have talked recently about two good families of spaces, CW complexes and manifolds. How are they related? A CW complex is a much more general kind of space. For instance, $S^1 \vee S^1$ has a perfectly good CW structure with a single 0-cell and two 1-cells, but it is not a manifold since the basepoint does not have a Euclidean neighborhood. On the other hand, most manifolds do admit CW decompositions.

Theorem 44.6 (Lee, 5.25)**.** *Every 1-manifold admits a (nice) CW decomposition.*

Theorem 44.7 (Lee, 5.36, 5.37)**.** *Every n-manifold admits a (nice) CW decomposition for $n = 2, 3$.*

According to p. 529 of Allen Hatcher's Algebraic Topology book, it is an open question whether or not every 4-manifold admits a CW decomposition. But n-manifolds for $n \geq 5$ do always admit a CW decomposition.

Another important problem, back purely in the realm of manifolds, is to try to list all manifolds of a given dimension.

Theorem 44.8 (Classification of 1-manifolds)**.** *Every nonempty, connected 1-manifold M is homeomorphic to S^1 if it is compact and to $\mathbb{R}$ if it is noncompact.*

For this theorem, it will be convenient to work with nice CW structures.

Definition 44.9. If X is a space with a CW structure, we say that an n-cell e_i^n is **regular** if the characteristic map $\Phi_i : D^n \longrightarrow \overline{e_i} \subset X$ is a homeomorphism onto its image. We say that a CW complex is regular if every cell is regular.

Proof. The first step is to show that every 1-manifold has a regular CW decomposition. The main idea is to cover M by a countable collection $\{U_n\}$ of regular charts (each closure $\overline{U_n}$ in M should be homeomorphic to $[0, 1]$). Then, using induction, it is possible to put a regular CW structure on $\mathcal{U}_n = \bigcup_{k=1}^n U_k$ in such a way that $\mathcal{U}_n \subseteq \mathcal{U}_{n+1}$ is the inclusion of a subcomplex. (See Lee 5.25 for more details.) Clearly, each 1-cell bounds two 0-cells, since the 1-cell is assumed to be regular. Somewhat less clear is the fact that each 0-cell is in the boundary of two 1-cells (see Lee 5.26).

We enumerate the 0-cells (aka vertices) and 1-cells (aka edges) in the following way. First, pick some 0-cell, and call it v_0. Pick an edge ending at v_0, and call this e_0. The other endpoint of e_0 we call v_1. The other edge ending at v_1 is called e_1. We can continue in this way to get $v_2, v_3, \dots$ and $e_2, e_3, \dots$. Now there is also another edge ending at v_0, which should be called e_{-1}. Let v_{-1} be the other endpoint. We can continue to get $v_{-2}, v_{-3}, \dots$ and $e_{-2}, e_{-3}, \dots$.

There are two cases to consider:

Case 1: The vertices v_i, $i \in \mathbb{Z}$ are all distinct. Then for each $n \in \mathbb{Z}$, we have an embedding $[n, n+1] \cong [-1, 1] \xrightarrow{\Phi_n} \overline{e_n^1}$. These glue together to give a continuous map $f : \mathbb{R} \longrightarrow X$. Our assumption means that f is injective when restricted to $\mathbb{Z}$. We can then see it is globally injective since its restriction to any $(n, n+1)$ is a characteristic map for a cell (thus injective) and all cells are disjoint.

Next, we show that f is open. Any open subset of $(n, n+1)$ is taken by f to an open subset of M, since the top-dimension cells are always open in a CW complex. It remains to show that f takes intervals of the form $(n - \epsilon, n + \epsilon)$ to open subsets of M. By taking ϵ small enough, we can ensure that this image is contained in (the closure of) two 1-cells. We can then see that this subset of M is open by pulling back along the characteristic maps (pulling back along these two characteristic maps will give half-open intervals in D^1).

Since M is connected, in order to show that f is surjective, it now suffices to show that $f(\mathbb{R})$ is closed. Let $x \notin f(\mathbb{R})$. If x lies in a 1-cell e, then e is a neighborhood of x disjoint from $f(\mathbb{R})$. The other possibility is that x is a 0-cell. But then x must be the endpoint of two closed 1-cells $\overline{e}$ and $\overline{e'}$. Neither e nor e' can be in $f(\mathbb{R})$ since this would imply that x also lies in $f(\mathbb{R})$. But then $e \cup \{x\} \cup e'$ is a neighborhood of x disjoint from $f(\mathbb{R})$.

We have shown that $f : \mathbb{R} \longrightarrow M$ is an open, continuous bijection. So it is a homeomorphism.

Case 2: For some $n \in \mathbb{Z}$ and $k > 0$, we have $v_n = v_{n+k}$. We may then pick n and k so that k is minimal. Then the vertices $v_n, \dots, v_{n+k-1}$ are distinct, as are the edges $e_n, \dots, e_{n+k-1}$. This implies that the restriction of f to $[n, n+k)$ is injective. If we consider the restriction only to the closed interval $[n, n+k]$, then we get a closed map, since the domain is compact and the target is

Hausdorff. We claim also that $f([n,n+k])$ is open in M. Indeed, if we pick any $x \in [n,n+k]$ which lies in an interval $(i,i+1)$, then the open 1-cell e_i is a neighborhood of $f(x)$ that is contained in the image of f. If we consider any interior integer $n < i < n+k$, then $e_{i-1} \cup \{v_i\} \cup e_i$ is an open neighborhood in the image of f. Finally, $e_n \cup \{n\} e_{n+k-1}$ is a neighborhood of $f(n) = f(n+k)$ in M.

Since the image $f([n,n+k])$ is both closed and open in M and M is connected, we conclude that $f([n,n+k]) = M$. Since $f(n) = f(n+k)$, we get an induced map

$$\overline{f} : [n,n+k]/\sim \; \cong S^1 \longrightarrow M$$

which is a bijection. Since S^1 is compact and M is Hausdorff, this is a homeomorphism. ■

We previously also briefly mentioned the idea of a "manifold with boundary". There is a similar result:

Theorem 44.10. *Every nonempty, connected* 1*-manifold with boundary is homeomorphic to* $[0,1]$ *if it is compact and to* $[0,1)$ *if it is noncompact.*

Next semester, we will similarly classify all compact 2-manifolds (the list of answers will be a little longer).

A closely related idea to CW complex is the notion of **simplicial complex**. A simplicial complex is built out of "simplices". By definition, an n**-simplex** is the convex hull of $n+1$ "affinely independent" points in $\mathbb{R}^k$, for $k \geq n+1$. This means that after translating this set so that one point moves to the origin, the resulting collection of points is linearly independent.

There is a standard n-simplex $\Delta^n \subseteq \mathbb{R}^{n+1}$ defined by

$$\Delta^n = \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid \sum t_i = 1, t_i \geq 0\}.$$

In general, if σ is an n-simplex generated by $\{t_0, \ldots t_n\}$, then the convex hull of any subset is called a **face** of the simplex. A (Euclidean) simplicial complex is then a subset of $\mathbb{R}^k$ that is a union of simplices such that any two overlapping simplices meet in a face of each. We also usually require the collection of simplices to be locally finite.

Since an n-simplex is homeomorphic to D^n, it can be seen that a simplicial complex is a regular CW complex. A decomposition of a manifold as a simplicial complex is known as a **triangulation** of the manifold. Just as one can ask about CW structures on manifolds, one can also ask about triangulations for manifolds.

Theorem 44.11.
(1) *Every* 1*-manifold is triangulable (indeed, we know the complete list of connected* 1*-manifolds).*
(2) *Tibor Radó proved in 1925 that every* 2*-manifold is triangulable.*
(3) *Edwin Moise proved in the 1950s that every* 3*-manifold is triangulable.*
(4) *Michael Freedman discovered the* 4*-dimensional* E_8*-manifold in 1982, which is not triangulable.*
(5) *Ciprian Manolescu showed in March 2013 that manifolds in dimension* ≥ 5 *are not triangulable.*