Math 225 - Practice Midterm 2 - SOLUTIONS

1. **True or False**?

   (a) The set of vectors $\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$ is a basis for $\mathbb{R}^3$.

   FALSE. This collection of vectors was shown to be *dependent* in solution 1(b) for the first practice exam. Since it is dependent, it cannot be a basis.

   (b) The set of vectors $\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$ is a basis for $\mathbb{R}^3$.

   TRUE. This collection of vectors was shown to be *independent* in solution 1(c) for the first practice exam. Since it is independent, it can be completed to a basis for $\mathbb{R}^3$. Since it already has 3 vectors and $\mathbb{R}^3$ is 3-dimensional, it must already be a basis.

   (c) If U is an echelon form for the matrix A, then the pivot columns of U form a basis for $C(A)$.

   FALSE. The pivot columns of the *original matrix* A form a basis for $C(A)$. The columns of U may not even be contained in $C(A)$.

   (d) If U is an echelon form for the matrix A, then the nonzero rows of U form a basis for $R(A)$.

   TRUE. The rows of U are linear combinations of the rows of A, so they are contained in $R(A)$. By the definition of echelon form, these rows must be independent. Finally, to see that they span $R(A)$, note that the row reduction steps can all be reversed, thereby expressing the rows of A as linear combinations of the rows of U.

   (e) If A is an $n \times n$ matrix, then $\det(-A) = -\det(A)$.

   FALSE. $\det(-A) = (-1)^n \det(A)$.

2. (a) Use Cramer's Rule to solve the matrix equation $\begin{pmatrix} 5 & 1 \\ 7 & 3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 13 \\ 15 \end{pmatrix}$.

   SOLUTION. We will need to know $\begin{vmatrix} 5 & 1 \\ 7 & 3 \end{vmatrix} = 15 - 7 = 8$. Cramer's Rule says that

$$x_1 = \frac{1}{8} \begin{vmatrix} 13 & 1 \\ 15 & 3 \end{vmatrix} = \frac{1}{8}(39 - 15) = \frac{1}{8}(24) = 3$$

   and

$$x_2 = \frac{1}{8} \begin{vmatrix} 5 & 13 \\ 7 & 15 \end{vmatrix} = \frac{1}{8}(75 - 91) = \frac{1}{8}(-16) = -2$$

   (b) Use Cramer's Rule to find the entry in position $(1, 2)$ of the *inverse* of $A = \begin{pmatrix} 2 & 1 & 5 \\ -1 & 6 & 2 \\ 1 & 3 & -2 \end{pmatrix}$.

SOLUTION. We first find $\det(A)$.

$$\begin{vmatrix} 2 & 1 & 5 \\ -1 & 6 & 2 \\ 1 & 3 & -2 \end{vmatrix} = - \begin{vmatrix} 1 & 3 & -2 \\ -1 & 6 & 2 \\ 2 & 1 & 5 \end{vmatrix} = - \begin{vmatrix} 1 & 3 & -2 \\ 0 & 9 & 0 \\ 0 & -5 & 9 \end{vmatrix} = - \begin{vmatrix} 9 & 0 \\ -5 & 9 \end{vmatrix} = -81$$

The required entry is then given by

$$\frac{1}{-81} \begin{vmatrix} 0 & 1 & 5 \\ 1 & 6 & 2 \\ 0 & 3 & -2 \end{vmatrix} = \frac{1}{81} \begin{vmatrix} 1 & 5 \\ 3 & -2 \end{vmatrix} = \frac{1}{81}(-2-15) = \frac{-17}{81}$$

3. (a) For which value of c does the equation $2x - y + 3z = c$ define a subspace of $\mathbb{R}^3$?

   SOLUTION. Any subspace must contain the origin, so c must be 0.

   (b) Find a basis for this subspace of $\mathbb{R}^3$. What is the dimension of this subspace?

   SOLUTION. We can think of this as the null space of the matrix $\begin{pmatrix} 2 & -1 & 3 \end{pmatrix}$. The variables y and z are free, and we find basis vectors $\begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -3/2 \\ 0 \\ 1 \end{pmatrix}$. The dimension is 2.

4. Consider the bases

$$\mathscr{B} = \left\{ \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right\}, \qquad \mathscr{C} = \left\{ \begin{pmatrix} 2 \\ -3 \end{pmatrix}, \begin{pmatrix} 1 \\ 5 \end{pmatrix} \right\}$$

for $\mathbb{R}^2$.

   (a) Find the change-of-basis matrix $\underset{\mathscr{C}\leftarrow\mathscr{B}}{P}$.

   SOLUTION. There are several possible approaches. The columns of $\underset{\mathscr{C}\leftarrow\mathscr{B}}{P}$ are the vectors of $\mathscr{B}$, written in terms of $\mathscr{C}$. Another approach is to use that

$$\underset{\mathscr{C}\leftarrow\mathscr{B}}{P} = \underset{\mathscr{C}\leftarrow\mathscr{E}}{P}\,\underset{\mathscr{E}\leftarrow\mathscr{B}}{P} = \left(\underset{\mathscr{E}\leftarrow\mathscr{C}}{P}\right)^{-1} \underset{\mathscr{E}\leftarrow\mathscr{B}}{P} = \begin{pmatrix} 2 & 1 \\ -3 & 5 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 2 \\ 4 & 2 \end{pmatrix}$$

$$= \frac{1}{13} \begin{pmatrix} 5 & -1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 4 & 2 \end{pmatrix} = \frac{1}{13} \begin{pmatrix} 1 & 8 \\ 11 & 10 \end{pmatrix}$$

   (b) Let $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Find $[\mathbf{v}]_{\mathscr{B}}$ and $[\mathbf{v}]_{\mathscr{C}}$.

   SOLUTION.

$$[\mathbf{v}]_{\mathscr{B}} = \underset{\mathscr{B}\leftarrow\mathscr{E}}{P}\mathbf{v} = \begin{pmatrix} 1 & 2 \\ 4 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{-6} \begin{pmatrix} 2 & -2 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{-6} \begin{pmatrix} 2 \\ -4 \end{pmatrix} = \begin{pmatrix} -1/3 \\ 2/3 \end{pmatrix}.$$

   It follows that

$$[\mathbf{v}]_{\mathscr{C}} = \underset{\mathscr{C}\leftarrow\mathscr{B}}{P}[\mathbf{v}]_{\mathscr{B}} = \frac{1}{13} \begin{pmatrix} 1 & 8 \\ 11 & 10 \end{pmatrix} \begin{pmatrix} -1/3 \\ 2/3 \end{pmatrix} = \frac{1}{13} \begin{pmatrix} 5 \\ 3 \end{pmatrix}.$$

5. Find all eigenvalues and a basis for each eigenspace for the following matrices. Use this to diagonalize these matrices.

(a) $A = \begin{pmatrix} 2 & 2 \\ 3 & 1 \end{pmatrix}$.

SOLUTION. The characteristic polynomial is $\det(A - \lambda I) = \lambda^2 - 3\lambda - 4$. This splits as $(\lambda - 4)(\lambda + 1)$, so the eigenvalues are 4 and -1. For $\lambda = -1$, we have

$$A + I = \begin{pmatrix} 3 & 2 \\ 3 & 2 \end{pmatrix} \sim \begin{pmatrix} 3 & 2 \\ 0 & 0 \end{pmatrix}.$$

An eigenvector is given by $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$.

For $\lambda = 4$, we have

$$A - 4I = \begin{pmatrix} -2 & 2 \\ -1 & 1 \end{pmatrix} \sim \begin{pmatrix} -2 & 2 \\ 0 & 0 \end{pmatrix}.$$

An eigenvector is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. We now have the diagonalization

$$A = \begin{pmatrix} 2 & 1 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -3 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & 1 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix} \frac{1}{5} \begin{pmatrix} 1 & -1 \\ 3 & 2 \end{pmatrix}.$$

(b) $B = \begin{pmatrix} 4 & 4 & -2 \\ 0 & 0 & 0 \\ 4 & 4 & -2 \end{pmatrix}$.

SOLUTION. The characteristic polynomial is

$$\det(B - \lambda I) = \begin{vmatrix} 4-\lambda & 4 & -2 \\ 0 & -\lambda & 0 \\ 4 & 4 & -2-\lambda \end{vmatrix} = -\lambda \begin{vmatrix} 4-\lambda & -2 \\ 4 & -2-\lambda \end{vmatrix} = -\lambda(\lambda^2 - 2\lambda) = -\lambda^2(\lambda - 2).$$

For the eigenvalue $\lambda = 2$, we have

$$B - 2\lambda = \begin{pmatrix} 2 & 4 & -2 \\ 0 & -2 & 0 \\ 4 & 4 & -4 \end{pmatrix} \sim \begin{pmatrix} 2 & 0 & -2 \\ 0 & 1 & 0 \\ 4 & 0 & -4 \end{pmatrix} \sim \begin{pmatrix} 2 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We find an eigenvector of $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. For the eigenvalue $\lambda = 0$, we have $B \sim \begin{pmatrix} 4 & 4 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

We find eigenvectors $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1/2 \\ 0 \\ 1 \end{pmatrix}$. A diagonalization for B is then

$$B = \begin{pmatrix} 1 & -1 & 1/2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1/2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}^{-1}.$$