Tuesday, October 25 ∗∗ *Solution sets, Linear Independence*

1. Consider the system of equations

$$\begin{aligned} 4x - y - 3z &= 10 \\ 2x + y - 2z &= 6 \\ -2x + 2y + z &= -4 \end{aligned}$$

   (a) Find the a particular solution and also the form of the general solution.

   (b) Find the form of the general solution to the corresponding homogeneous system. (Hint: you shouldn't have to work very hard.)

2. Consider the following matrices.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}, C = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 6 & 2 & -4 \\ 2 & -1 & -1 & 4 \end{pmatrix}, D = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 6 & -1 \\ 0 & 2 & -1 \\ 1 & -4 & 4 \end{pmatrix}.$$

   (a) From the shape *alone*, which matrices cannot have independent columns?

   (b) Which matrices have independent columns?

   (c) If the columns of a matrix are independent, which variables in the corresponding homogeneous system are "basic" variables (aka pivot variables) and which variables are free variables?

3. Suppose that $\mathbf{u}$ and $\mathbf{v}$ are independent. Let $\mathscr{P}$ be the plane $\mathscr{P} = \text{Span}\{\mathbf{u}, \mathbf{v}\}$. Show that any collection $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ of three vectors in $\mathscr{P}$ must be dependent.

4. The largest number of linearly independent columns of a matrix A is called the **column rank** of A.

   (a) Explain why the column rank is unchanged by any row operation. In particular, this says that a matrix A has the same column rank as that of its reduced row echelon form U.

   (b) If A is a 4×3 matrix such that the matrix equation $A\mathbf{x} = \mathbf{b}$ has a *single* solution, what is the column rank of A?