**Tuesday, November 29** $\quad **\quad$ *Diagonalization & the Dot Product*

1. Suppose that an $n \times n$ matrix $A$ has only the eigenvalue 1 and that $A$ has $n$ independent eigenvectors (all for the eigenvalue 1). What matrix is $A$? Justify your answer.

2. (Generalized eigenvectors) Let

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 4 & 1 \\ -2 & -8 & -2 \end{pmatrix}$$

   (a) Find the eigenvalues of $A$ and find a basis for each eigenspace.

   (b) The matrix $A$ is *not* diagonalizable. One of the eigenvalues does not have "enough" eigenvectors. Call this eigenvalue $\lambda_1$. The dimension of the nullspace of $A - \lambda_1 I$ is smaller than expected.

   One can look for "generalized eigenvectors" by considering the nullspace of powers of the matrix $A - \lambda_1 I$. For instance, if $(A - \lambda_1 I)^2 \mathbf{w} = \mathbf{0}$, then $(A - \lambda_1 I)\mathbf{w}$ is an eigenvector. Let $\mathbf{v}_1$ be the eigenvector you found in part (a), and solve the matrix equation

$$(A - \lambda_1 I)\mathbf{w} = \mathbf{v}_1.$$

   (c) Use part (b) to show that $A$ is similar to a matrix of the form

$$\begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}.$$

   This is called the "Jordan canonical form" for the nondiagonalizable matrix $A$.

3. Find the magnitude of $\mathbf{v} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$ and find a unit vector $\mathbf{u}$ pointing in the same direction as $\mathbf{v}$.

4. Show that $\mathbf{u} \cdot \mathbf{v} = 0$ if and only if $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$.