Tuesday, December 6 ** *Orthogonal bases & Orthogonal projections* ** **Solutions**

1. A basis $\mathscr{B}$ is called an **orthonormal** basis if it is orthogonal and each basis vector has norm equal to 1.

   (a) Convert the orthogonal basis

   $$\mathscr{B} = \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

   into an orthonormal basis $\mathscr{C}$.

   Solution. We just scale each basis vector by its length. The new, orthonormal, basis is

   $$\mathscr{C} = \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

   (b) Find the coordinates of the vectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} -4 \\ 2 \\ 7 \end{pmatrix}$ in the basis $\mathscr{C}$.

   Solution. The formula is the same as for a general orthogonal basis: writing $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ for the basis vectors, the formula for each coordinate of $\mathbf{v}_1$ in the basis $\mathscr{C}$ is

   $$\frac{\mathbf{u}_i \cdot \mathbf{v}_1}{\|\mathbf{u}_i\|^2} = \mathbf{u}_i \cdot \mathbf{v}_1.$$

   We compute to find

   $$\mathbf{u}_1 \cdot \mathbf{v}_1 = \frac{2}{\sqrt{2}} = \sqrt{2}, \qquad \mathbf{u}_2 \cdot \mathbf{v}_1 = \frac{-6}{\sqrt{6}} = -\sqrt{6}, \qquad \mathbf{u}_3 \cdot \mathbf{v}_1 = \frac{9}{\sqrt{3}} = 3\sqrt{3},$$

   so $(\mathbf{v}_1)_{\mathscr{C}} = \begin{pmatrix} \sqrt{2} \\ -\sqrt{6} \\ 3\sqrt{3} \end{pmatrix}$. Similarly, we get

   $$\mathbf{u}_1 \cdot \mathbf{v}_2 = \frac{6}{\sqrt{2}} = 3\sqrt{2}, \qquad \mathbf{u}_2 \cdot \mathbf{v}_2 = \frac{-16}{\sqrt{6}} = -8\sqrt{\frac{2}{3}}, \qquad \mathbf{u}_3 \cdot \mathbf{v}_2 = \frac{5}{\sqrt{3}},$$

   so $(\mathbf{v}_2)_{\mathscr{C}} = \begin{pmatrix} 3\sqrt{2} \\ -8\sqrt{2/3} \\ 5/\sqrt{3} \end{pmatrix}$.

   Orthonormal bases are *very* convenient for calculations.

2. (The Gram-Schmidt process) There is a standard process for converting a basis into an orthogonal basis. Let $\mathscr{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a basis for $\mathbb{R}^3$. In this problem, you will find an orthogonal basis $\mathscr{C} = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$.

   Start by setting $\mathbf{w}_1 = \mathbf{v}_1$. Then we want $\mathbf{w}_2$ to be orthogonal to $\mathbf{v}_1$. If we write $\mathbf{p}$ for the projection of $\mathbf{v}_2$ onto $\mathbf{w}_1$, then $\mathbf{v}_2 - \mathbf{p}$ is orthogonal to $\mathbf{w}_1$, so we may choose this for $\mathbf{w}_2$. In other words,

$$\mathbf{w}_2 = \mathbf{v}_2 - \operatorname{proj}_{\mathbf{w}_1}(\mathbf{v}_2).$$

   Next, we want $\mathbf{w}_3$ to be orthogonal to **both** $\mathbf{w}_1$ and $\mathbf{w}_2$, so we define

$$\mathbf{w}_3 = \mathbf{v}_3 - \operatorname{proj}_{\mathbf{w}_1}(\mathbf{v}_3) - \operatorname{proj}_{\mathbf{w}_2}(\mathbf{v}_3).$$

   (a) Use the Gram-Schmidt process to convert

$$\mathscr{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \right\}$$

   into an orthogonal basis $\mathscr{C}$.

   Solution. We set $\mathbf{w}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. Then

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{w}_1}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} - \frac{3}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 1/2 \\ 1 \end{pmatrix}.$$

   We then take

$$\mathbf{w}_3 = \mathbf{v}_3 - \frac{\mathbf{v}_3 \cdot \mathbf{w}_1}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \frac{\mathbf{v}_3 \cdot \mathbf{w}_2}{\|\mathbf{w}_2\|^2}\mathbf{w}_2 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} - \frac{2}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \frac{2}{3/2}\begin{pmatrix} -1/2 \\ 1/2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/3 \\ 1/3 \\ -1/3 \end{pmatrix}.$$

   (b) Convert this orthogonal basis into an orthonormal basis, and then find the coordinates of the vector $\mathbf{v} = \begin{pmatrix} 7 \\ 2 \\ 1 \end{pmatrix}$ in this orthonormal basis.

   Solution. The orthonormal basis is

$$\mathscr{C} = \left\{ \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \sqrt{\frac{2}{3}}\begin{pmatrix} -1/2 \\ 1/2 \\ 1 \end{pmatrix}, \sqrt{3}\begin{pmatrix} -1/3 \\ 1/3 \\ -1/3 \end{pmatrix} \right\}.$$

   Note that another way to write this same basis is as

$$\mathscr{C} = \left\{ \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{6}}\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}, \frac{1}{\sqrt{3}}\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \right\}.$$

A computation as in problem 1 gives $(\mathbf{v})_{\mathscr{C}} = \begin{pmatrix} 9/\sqrt{2} \\ -\sqrt{3/2} \\ -2\sqrt{3} \end{pmatrix}$.

3. Find the least squares solution to the system of equations

$$\begin{aligned} 2x + y &= 3 \\ -x - y &= 2 \\ 3x + y &= 3. \end{aligned}$$

Solution. We have $A = \begin{pmatrix} 2 & 1 \\ -1 & -1 \\ 3 & 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}$. The least squares solution is the solution to the normal equation

$$A^T A\mathbf{x} = A^T\mathbf{b}.$$

The matrix $A^T A$ is

$$A^T A = \begin{pmatrix} 14 & 6 \\ 6 & 3 \end{pmatrix}$$

with inverse

$$(A^T A)^{-1} = \frac{1}{6}\begin{pmatrix} 3 & -6 \\ -6 & 14 \end{pmatrix}.$$

The least squares solution is therefore given by

$$\mathbf{x} = (A^T A)^{-1} A^T\mathbf{b} = \frac{1}{6}\begin{pmatrix} 3 & -6 \\ -6 & 14 \end{pmatrix}\begin{pmatrix} 13 \\ 4 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 15 \\ -22 \end{pmatrix}.$$

Note that

$$A\mathbf{x} = \begin{pmatrix} 4/3 \\ 7/6 \\ 23/6 \end{pmatrix} \neq \mathbf{b}$$

so the least squares solution $\mathbf{x}$ is not an *actual* solution.