Worksheet 1 solution : Tuesday, August 23

1. (a) $\frac{d}{dt}(h(t)) = \frac{d}{dt}(\sin(\cos(\tan t)))$

$= \cos(\cos(\tan t))\frac{d}{dt}(\cos(\tan t))$

$= \cos(\cos(\tan t))(-\sin(\tan t))\frac{d}{dt}(\tan t)$

$= \cos(\cos(\tan t))(-\sin(\tan t))\sec^2 t$

1. (b) From $\frac{ds}{dt} = \frac{ds}{dx}\cdot\frac{dx}{dt}$, we get

$$\frac{ds}{dt} = \frac{1}{4x^{3/4}} \cdot \frac{f'(t)}{f(t)}$$

But we need to make sure that $\frac{ds}{dt}$ is a single variable function of f,

So $\frac{ds}{dt} = \frac{1}{4[\ln(f(t))]^{3/4}} \cdot \frac{f'(t)}{f(t)}$.

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2. (a) Note that $\left(\frac{x}{5}\right)^2 + \left(\frac{y}{3}\right)^2 = \sin^2(3t) + \cos^2(3t) = 1$. So this parametrizes (at least part of) the ellipse $\left(\frac{x}{5}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$.

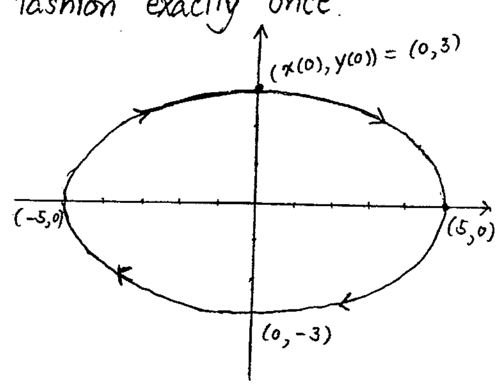
By examining differing values of t in $0 \leq t < \frac{2\pi}{3}$, we see that this parametrization travels the ellipse in a clockwise fashion exactly once.

$t = 0$: $(x(0), y(0)) = (0, 3)$

$t = \pi/6$: $(x(\pi/6), y(\pi/6)) = (5, 0)$

$t = \pi/3$: $(x(\pi/3), y(\pi/3)) = (0, -3)$

$t = \pi/2$: $(x(\pi/2), y(\pi/2)) = (-5, 0)$

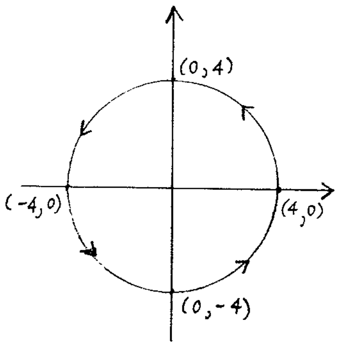


If we let t vary between 0 and 2π, we will traverse the ellipse 3 times.

2.(b) Arc length $s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$$= \int_0^{2\pi/3} \sqrt{(15\cos(3t))^2 + (-9\sin(3t))^2}\, dt .$$

2.(c)



If we let $x = 4\cos t$ and $y = 4\sin t$, then $x^2+y^2 = (4\cos t)^2 + (4\sin t)^2 = 16$.
Moreover, as t increases, this parametrization traverses the circle in a counterclockwise fashion:

$t=0$: $(x(0), y(0)) = (4,0)$

$t = \pi/2$: $(x(\frac{\pi}{2}), y(\frac{\pi}{2})) = (0,4)$

$t = \pi$: $(x(\pi), y(\pi)) = (-4,0)$

$t = 3\pi/2$: $\left(x\left(\frac{3\pi}{2}\right), y\left(\frac{3\pi}{2}\right)\right) = (0,-4)$

$t = 2\pi$: $(x(2\pi), y(2\pi)) = (4,0)$.

To ensure that we travel the curve only once, we restrict t to the interval $[0, 2\pi)$.
So the parametrization is $\begin{cases} x = 4\cos t \\ y = 4\sin t \end{cases}$, when $0 \le t < 2\pi$.

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3. (a) First, we find the critical points of $f(x)$.

$$f'(x) = 4x^3 - 16x$$

$f'(x) = 0$ when $4x^3 - 16x = 0$

$$4x(x^2-4) = 0$$

$$4x(x-2)(x+2) = 0 .$$

Hence $f'(x) = 0$ when $x = 0$, $x = 2$, or $x = -2$

Now apply the 2nd Derivative Test to the three critical points:

From $f''(x) = 12x^2 - 16$, we get

$f''(0) = -16 < 0$, so $y = f(x)$ is concave down at the point $(0, f(0))$.
So a local max occurs at $(0, 10)$.

$f''(-2) = 32 > 0$, so $y = f(x)$ is concave up at the point $(-2, f(-2))$.
A local min occurs at $(-2, -6)$

$f''(2) = 32 > 0$, so $y = f(x)$ is concave up at the point $(2, f(2))$.
A local min occurs at $(2, -6)$.

3. (b) First, find the critical points of $h(s)$.

$$h'(s) = 4s^3 + 12s^2.$$

Then $h'(s) = 0$ when $4s^3 + 12s^2 = 0$

$$4s^2(s+3) = 0.$$

So $h'(s) = 0$ when $s = 0$ and $s = -3$.

For the 1st Derivative Test, we need to determine if h is increasing or decreasing on the intervals $(-\infty, -3)$, $(-3, 0)$, and $(0, \infty)$.

On $(-\infty, -3)$ choose any test point (for example, choose $s = -1000$). The sign of $h'(s) = 4s^3 + 12s^2 < 0$ on this interval. Hence $h(s)$ is decreasing on $(-\infty, -3)$.

On $(-3, 0)$ choose any test point (e.g. choose $s = -1$). The sign of $h'(s) = 4s^3 + 12s^2 > 0$ on this interval. Hence $h(s)$ is increasing on $(-3, 0)$.

On $(0, \infty)$ choose any test point (e.g. choose $s = 1000$). The sign of $h'(s) = 4s^3 + 12s^2 > 0$ on this interval. Hence $h(s)$ is increasing on $(0, \infty)$.

Since at $s=-3$ the function changes from decreasing to increasing, the function must have obtained a local min at $s=-3$.

At $s=0$, neither a max or a min occurs in the value of h.

3.(c) When $s=-3$, $h''(-3) = 36 > 0$. A local min occurs when $s=-3$ by the 2nd Derivative Test.

When $s=0$, $h''(0)=0$. The 2nd Derivative Test is inconclusive. The graph of $y=h(s)$ has no concavity at $(0, h(0))$. Without more information (the 1st Derivative Test), we are unable to identify $(0, h(0))$ as a local max, min, or a point of inflection.

4.(a) Recall that in Calc I and II, the "best linear approximation" is synonymous with the equation of the tangent line or the 1st-order Taylor polynomial.

Here, $f'(x) = 2xe^{-x} + x^2(-e^{-x})$.

Since $f'(0)=0$, the tangent line has no slope at $(0, f(0)) = (0,0)$.

The equation of the tangent line is $y=0$.

4.(b) By definition, the second-order Taylor polynomial at $x=0$ is

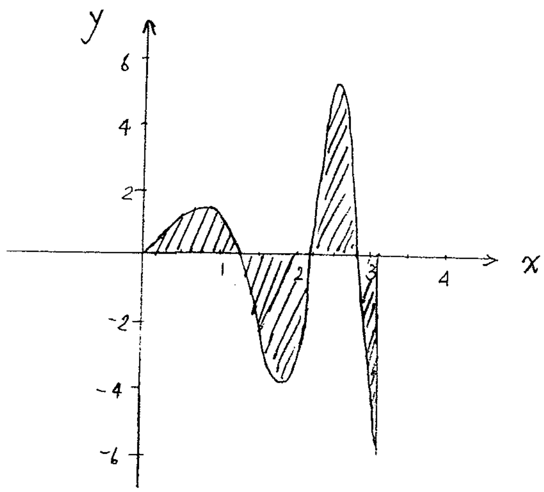
$$T_2(x) = f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2.$$

Since $f''(x) = 2e^{-x} - 4xe^{-x} + x^2e^{-x}$, we compute that $f''(0)=2$.

Hence $T_2(x) = 0 + \frac{0}{1!}(x-0) + \frac{2}{2!}(x-0)^2 = x^2$.

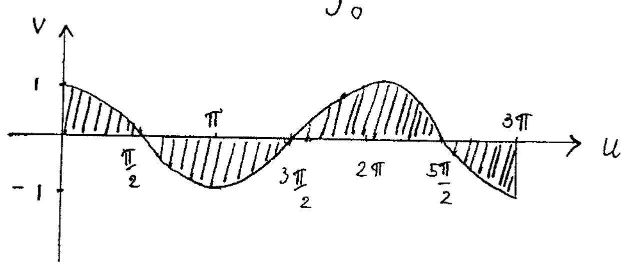
4. (c) The second-order Taylor polynomial is the best quadratic approximation to the curve $y = f(x)$ at the point $(0, f(0))$. Since $T_2(x) = x^2$ clearly has a local minimum at $(0,0)$, and $(0,0)$ is the location of a critical point of f, then f must also have a local minimum at $(0,0)$.

5. (a)



5. (b) Let $u = x^2$. Then $du = 2x\,dx$, so the integral becomes

$$\int_0^{\sqrt{3\pi}} 2x \cos(x^2)\,dx = \int_0^{3\pi} \cos u \, du.$$



5. (c)
$$\int_0^{\sqrt{3\pi}} 2x \cos(x^2)\,dx = \int_0^{3\pi} \cos u \, du$$
$$= \Big[\sin u\Big]_{u=0}^{u=3\pi}$$
$$= \sin 3\pi - \sin 0$$
$$= 0.$$