Tuesday, August 30 ∗∗ *Orthogonal projections and planes in* $\mathbb{R}^3$.

1. Let $\mathbf{u} = (2,4)$, $\mathbf{v} = (3,1)$, and $\mathbf{w} = (4,4)$.

   (a) Find the projections $\text{proj}_{\mathbf{v}}(\mathbf{u})$ and $\text{proj}_{\mathbf{v}}(\mathbf{w})$ of the vectors $\mathbf{u}$ and $\mathbf{w}$ onto $\mathbf{v}$.

   (b) The orthogonal complement of $\mathbf{u}$ with respect to $\mathbf{v}$ is the vector

   $$\text{orth}_{\mathbf{v}}(\mathbf{u}) = \mathbf{u} - \text{proj}_{\mathbf{v}}(\mathbf{u}).$$

   Find $\text{orth}_{\mathbf{v}}(\mathbf{u})$ and $\text{orth}_{\mathbf{v}}(\mathbf{w})$. Represent all seven vectors in a figure (the three originals, the two projections, and the two complements).

   (c) Check that the complements $\text{orth}_{\mathbf{v}}(\mathbf{u})$ and $\text{orth}_{\mathbf{v}}(\mathbf{w})$ are both orthogonal to $\mathbf{v}$.

2. Let $P = (1,2,3)$, $Q = (0,1,2)$, and $\mathbf{u} = (1,2,1)$.

   (a) Find the distance from P to the line $t\,\mathbf{u}$. Hint: What is the closest point to P on this line?

   (b) Find the distance from P to the line $Q + t\,\mathbf{u}$. Hint: The distance from P to $Q + t\,\mathbf{u}$ is the same as the distance from which point to the line $t\,\mathbf{u}$?

3. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^2$. Set

   $$\mathbf{w} = \text{orth}_{\mathbf{v}}(\mathbf{u}) = u - \text{proj}_{\mathbf{v}}(\mathbf{u}).$$

   Find $\text{proj}_{\mathbf{w}}\,\mathbf{u}$ in terms of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

4. Let $P = (-1,3,0)$ and let $\mathscr{P}$ be the plane described by the equation

   $$-2x + y + z = 3.$$

   (a) Find a normal vector $\mathbf{n}$ to the plane $\mathscr{P}$. Hint: First find a normal vector to the plane through the origin $-2x + y + z = 0$. How is this normal vector related to $\mathbf{n}$?

   (b) Find a point Q on the plane $\mathscr{P}$. Check that if R is also a point on $\mathscr{P}$, then $R - Q$ lies on the plane through the origin

   $$-2x + y + z = 0.$$

   In other words, this says that the translate of the plane $\mathscr{P}$ by Q is the plane through the origin.

   (c) Find the projection $\text{proj}_{\mathbf{n}}(P - Q)$.

   (d) Use the information from the previous parts to find the distance from P to $\mathscr{P}$.