Tuesday, August 30 - Orthogonal projections and planes in $\mathbb{R}^3$.

1. Let $\mathbf{u} = (2,4)$, $\mathbf{v} = (3,1)$, and $\mathbf{w} = (4,4)$.

   (a) Find the projections $\text{proj}_{\mathbf{v}}(\mathbf{u})$ and $\text{proj}_{\mathbf{v}}(\mathbf{w})$ of the vectors $\mathbf{u}$ and $\mathbf{w}$ onto $\mathbf{v}$.

   (b) The orthogonal complement of $\mathbf{u}$ with respect to $\mathbf{v}$ is the vector

   $$\text{orth}_{\mathbf{v}}(\mathbf{u}) = \mathbf{u} - \text{proj}_{\mathbf{v}}(\mathbf{u}).$$

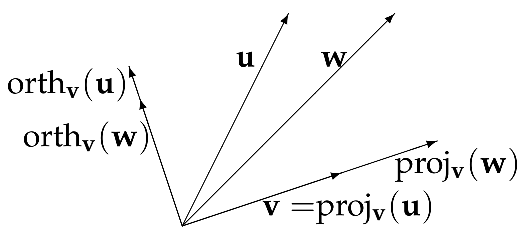
   Find $\text{orth}_{\mathbf{v}}(\mathbf{u})$ and $\text{orth}_{\mathbf{v}}(\mathbf{w})$. Represent all seven vectors in a figure (the three originals, the two projections, and the two complements).

   (c) Check that the complements $\text{orth}_{\mathbf{v}}(\mathbf{u})$ and $\text{orth}_{\mathbf{v}}(\mathbf{w})$ are both orthogonal to $\mathbf{v}$.

SOLUTION:

(a) Use the formula $\text{proj}_{\mathbf{v}}(\mathbf{u}) = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v}$. $\text{proj}_{\mathbf{v}}(\mathbf{u}) = \frac{10}{10}\mathbf{v} = \mathbf{v} = (3,1)$. $\text{proj}_{\mathbf{v}}(\mathbf{w}) = \frac{16}{10}\mathbf{v} = (24/5, 8/5)$.

(b) $\text{orth}_{\mathbf{v}}(\mathbf{u}) = (2,4) - (3,1) = (-1,3)$ and $\text{orth}_{\mathbf{v}}(\mathbf{w}) = (4,4) - (24/5,8/5) = (-4/5,12/5)$.



(c) $\text{orth}_{\mathbf{v}}(\mathbf{u}) \cdot \mathbf{v} = (-1,3) \cdot (3,1) = 0$ and $\text{orth}_{\mathbf{v}}(\mathbf{w}) \cdot \mathbf{v} = (-4/5,12/5) \cdot (3,1) = 0$ so both are orthogonal to $\mathbf{v}$.

2. Let $P = (1,2,3)$, $Q = (0,1,2)$, and $\mathbf{u} = (1,2,1)$.

   (a) Find the distance from P to the line $t\,\mathbf{u}$. Hint: What is the closest point to P on this line?

   (b) Find the distance from P to the line $Q + t\,\mathbf{u}$. Hint: The distance from P to $Q + t\,\mathbf{u}$ is the same as the distance from which point to the line $t\,\mathbf{u}$?

SOLUTION:

(a) The tip of the projection of the vector $\mathbf{P} = (1,2,3)$ onto $\mathbf{u}$ gives the closest point on the line $t\mathbf{u}$ to P. The length of the orhogonal complement of $\mathbf{P}$ with respect to $\mathbf{u}$ is the distance from the point P to the line $t\mathbf{u}$. We have $\text{proj}_{\mathbf{u}}(\mathbf{P}) = \left(\frac{\mathbf{u}\cdot\mathbf{P}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u} = 4/3\mathbf{u} = (4/3,8/3,4/3)$ and $\text{orth}_{\mathbf{u}}(\mathbf{P}) = \mathbf{P} - \text{proj}_{\mathbf{u}}(\mathbf{P}) = (1,2,3) - (4/3,8/3,4/3) = (-1/3,-2/3,5/3)$. So the distance from P to $t\mathbf{u}$ is $||(-1/3,-2/3,5/3)|| = \frac{\sqrt{30}}{3}$.

(b) The distance from P to $Q + t\mathbf{u}$ is the same as the distance from $P - Q$ to the line $t\mathbf{u}$. Let $\mathbf{v} = \mathbf{P} - \mathbf{Q} = (1,1,1)$. Then the distance from P to $Q + t\mathbf{u}$ is $||\text{orth}_{\mathbf{u}}(\mathbf{v})|| = ||(1,1,1) - \text{proj}_{\mathbf{u}}(\mathbf{v})|| = ||(1,1,1) - 2/3(1,2,1)|| = ||(1/3, -1/3, 1/3)|| = \sqrt{3}/3$.

3. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^2$. Set

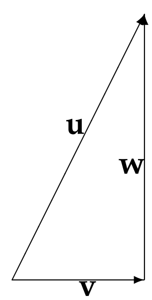
$$\mathbf{w} = \text{orth}_{\mathbf{v}}(\mathbf{u}) = u - \text{proj}_{\mathbf{v}}(\mathbf{u}).$$

Find $\text{proj}_{\mathbf{w}}\, \mathbf{u}$ in terms of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

SOLUTION:

$$\text{proj}_{\mathbf{w}}(\mathbf{u}) = \left(\frac{\mathbf{u}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}\right)\mathbf{w} = \left(\frac{\mathbf{u}\cdot\mathbf{u} - \frac{(\mathbf{u}\cdot\mathbf{v})^2}{\mathbf{v}\cdot\mathbf{v}}}{\mathbf{u}\cdot\mathbf{u} - 2\frac{(\mathbf{u}\cdot\mathbf{v})^2}{\mathbf{v}\cdot\mathbf{v}} + \frac{(\mathbf{u}\cdot\mathbf{v})^2}{\mathbf{v}\cdot\mathbf{v}}}\right)\mathbf{w} = \mathbf{w}$$

This is easier to see from the picture:



4. Let $P = (-1, 3, 0)$ and let $\mathscr{P}$ be the plane described by the equation

$$-2x + y + z = 3.$$

(a) Find a normal vector $\mathbf{n}$ to the plane $\mathscr{P}$. Hint: First find a normal vector to the plane through the origin $-2x + y + z = 0$. How is this normal vector related to $\mathbf{n}$?

(b) Find a point Q on the plane $\mathscr{P}$. Check that if R is also a point on $\mathscr{P}$, then $R - Q$ lies on the plane through the origin

$$2x + y + z = 0.$$

In other words, this says that the translate of the plane $\mathscr{P}$ by Q is the plane through the origin.

(c) Find the projection $\text{proj}_{\mathbf{n}}(P - Q)$.

(d) Use the information from the previous parts to find the distance from P to $\mathscr{P}$.

SOLUTION:

(a) A normal vector for the plane $-2x+y+z=0$ is given by $\mathbf{n}'=(-2,1,1)$ since the condition $-2x+y+z=0$ says that the plane consists of those points (x,y,z) such that $\mathbf{n}'\cdot(x,y,z)=0$ and this is the same as saying that $\mathbf{n}'$ is orthogonal to the position vector of the point (x,y,z). Since the plane passes through the origin, being orthogonal to the position vector of every point in the plane is equivalent to being orthogonal to the plane. Now observe that the plane with equation $-2x+y+z=3$ can be obtained from the plane $-2x+y+z=0$ by shifting the latter 3 units up. It follows that the vector $\mathbf{n}'$ is also a normal vector for the plane $-2x+y+z=3$, so we can take $\mathbf{n}=\mathbf{n}'=(-2,1,1)$.

(b) Setting $x=y=0$ in the equation $-2x+y+z=3$ we see that $Q=(0,0,3)$ is a point on the plane $\mathscr{P}$. Suppose $R=(a,b,c)$ is another point on $\mathscr{P}$ so $-2a+b+c=3$. Then $R-Q=(a,b,c-3)$. We have $-2a+b+(c-3)=(-2a+b+c)-3=3-3=0$ since R is in $\mathscr{P}$. It follows that $R-Q$ is in the plane through the origin with equation $-2x+y+z=0$ for every R in $\mathscr{P}$, so the translate of $\mathscr{P}$ by Q is a plane through the origin.

(c) We have $P-Q=(-1,3,-3)$. Now

$$\operatorname{proj}_{\mathbf{n}}(P-Q)=\left(\frac{(-1,3,-3)\cdot(-2,1,1)}{(-2,1,1)\cdot(-2,1,1)}\right)(-2,1,1)=\frac{2}{6}(-2,1,1)=(-2/3,1/3,1/3).$$

(d) The distance from P to $\mathscr{P}$ is the length of the projection of $P-Q$ onto $\mathbf{n}$. We have $||\operatorname{proj}_{\mathbf{n}}(P-Q)||=||(-2/3,1/3,1/3)||=\frac{\sqrt{6}}{3}$.