Lecture 10

Sept. 14
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Last time Differentiable functions $f: \mathbb{R}^2 \to \mathbb{R}$.

Idea: f has good linear approximation at (c,d)

Precise definition $\lim_{(x,y)\to(c,d)} \frac{f(x,y) - L(x,y)}{\|(x,y)-(c,d)\|} = 0$,

where $L(x,y) = f(c,d) + \frac{\partial f}{\partial x}(c,d)(x-c) + \frac{\partial f}{\partial y}(c,d)(y-d)$.

Looking at example $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

Is this diff. at $(0,0)$?

Saw that $\underset{\text{linear approximation to } f \text{ at } (0,0)}{L(x,y)} = 0$.

Is 0 a good approx to f near $(0,0)$?

$$\lim_{(x,y)\to(0,0)} \frac{\frac{xy}{x^2+y^2} - 0}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{xy}{(x^2+y^2)^{3/2}}$$

Along lines $x=0$ or $y=0$, get 0.

Along line $x=y$, limit DNE. (Note $f(x,x) = \frac{1}{2}$ for any $x \neq 0$)

0 is bad approx. along line $y=x$.

So f is not diff @ $(0,0)$.

Saw $f = \frac{1}{2}$ on line $x=y$, but $f(0,0)=0$, so f not continuous at $(0,0)$.

Theorem If f is differentiable at (c,d), it is also continuous at (c,d).

So first step towards being differentiable is being continuous.

Why is theorem true?

Assume $$\lim_{(x,y)\to(c,d)} \frac{f(x,y) - \left[f(c,d) + \frac{\partial f}{\partial x}(x-c) + \frac{\partial f}{\partial y}(y-c)\right]}{\|(x,y)-(c,d)\|} = 0.$$

Then can multiply by $\|(x,y)-(c,d)\|$, limit still 0.

But $$\lim_{(x,y)\to(c,d)} f(x,y) - f(c,d) + \underbrace{\frac{\partial f}{\partial x}(x-c)}_{\to 0} + \underbrace{\frac{\partial f}{\partial y}(y-c)}_{\to 0}$$

So $\lim_{(x,y)\to(c,d)} f(x,y) - f(c,d) = 0$, or $\lim_{(x,y)\to(c,d)} f(x,y) = f(c,d)$.

So f is continuous at (c,d).

How do tell when f is differentiable?

Theorem If $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$ exist on disc around (c,d) & are continuous at (c,d), **then** f is differentiable at (c,d).

In example $f(x,y) = \frac{xy}{x^2+y^2}$,

$f_x(x,y) = \frac{y^3 - x^2 y}{(x^2+y^2)^2}$ $(x,y) \neq (0,0)$ and $f_x(0,0) = 0$.

What is $\lim_{(x,y)\to(0,0)} f_x(x,y)$?

Along line $x=0$, get $\lim_{y\to 0} \frac{y^3}{y^4} = \lim_{y \to 0} \frac{1}{y}$ DNE ③

So $f_x(x,y)$ not continuous at $(0,0)$.

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§14.5 Chain Rule

Old Chain Rule: f & g diff. functions (1 variable) then $(f\circ g)'(x) = f'(g(x))\cdot g'(x)$.

Describes derivative of composition

$$\mathbb{R} \xrightarrow{g} \mathbb{R} \xrightarrow{f} \mathbb{R}$$

Now let $f: \mathbb{R}^2 \to \mathbb{R}$ (or $\mathbb{R}^3 \to \mathbb{R}$).

If $g: \mathbb{R} \to \mathbb{R}$ & $h: \mathbb{R} \to \mathbb{R}$, can consider $F(t) = f(g(t), h(t))$. This a function $F: \mathbb{R} \to \mathbb{R}$.

<u>Chain Rule (Version I)</u> Assume f, g, h diff.

$$\frac{dF}{dt} = \frac{\partial f}{\partial x}\frac{dg}{dt} + \frac{\partial f}{\partial y}\frac{dh}{dt}$$

<u>Why?</u> At $t=c$, best linear approx to $F(t)$ is

$$f(g(c),h(c)) + \frac{dF}{dt}(c)(t-c).$$

But best linear approx to $f(-,-)$ at $g(c), h(c)$ is

$$f(g(c),h(c)) + \frac{\partial f}{\partial x}(g(c),h(c))(x-g(c)) + \frac{\partial f}{\partial y}(g(c),h(c))(y-h(c))$$

Plug in $x = g(t)$, $y = h(t)$ & use linear approximations

$$g(t) \sim g(c) + g'(c)(t-c), \quad h(t) \sim h(c) + h'(c)(t-c)$$

to get (NEXT PAGE)

$$f(g(c),h(c)) + \frac{\partial f}{\partial x}(g(c),h(c))\left([\cancel{g(c)}+g'(c)(t-c)] - \cancel{g(c)}\right)$$
$$+ \frac{\partial f}{\partial y}(g(c),h(c))\left([\cancel{h(c)}+h'(c)(t-c)] - \cancel{h(c)}\right)$$
$$= f(g(c),h(c)) + \left[\frac{\partial f}{\partial x}(g(c),h(c))\, g'(c) + \frac{\partial f}{\partial y}(g(c),h(c))\, h'(c)\right](t-c)$$

Comparing linear approximations for $F(t)$, find

$$\frac{dF}{dt}(c) = \frac{\partial f}{\partial x}(g(c),h(c))\,\frac{dg}{dt}(c) + \frac{\partial f}{\partial y}(g(c),h(c))\,\frac{dh}{dt}(c)$$

<u>Example</u> $f(x,y) = x^2+y^2$, $g(t)=t\cos t$, $h(t)=t\sin t$.

Then $F(t) = f(t\cos t, t\sin t) = t^2\cos^2 t + t^2\sin^2 t = t^2$.

So $F'(t) = 2t$.

<u>Chain Rule</u>: $F'(t) = \frac{\partial f}{\partial x}(t\cos t, t\sin t)\, g'(t) + \frac{\partial f}{\partial y}(t\cos t, t\sin t)\, h'(t)$

$$= 2t\cos t\,(\cos t - t\sin t) + 2t\sin t\,(\sin t + t\cos t) = 2t$$

<u>Now</u> suppose g & h of form $\mathbb{R}^2 \to \mathbb{R}$.

Then $F = f(g,h)$ is $\mathbb{R}^2 \to \mathbb{R}$.

What are $\frac{\partial F}{\partial s}$, $\frac{\partial F}{\partial t}$?

<u>Chain Rule (Version II)</u> Assume f, g, h diff.

Then
$$\frac{\partial F}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial h}{\partial s}$$

&
$$\frac{\partial F}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial h}{\partial t}.$$

Now can generalize easily:

given $f: \mathbb{R}^n \to \mathbb{R}$ diff

& n diff. functions $g_1: \mathbb{R}^k \to \mathbb{R}$

$g_2: \mathbb{R}^k \to \mathbb{R}$

$\vdots$

$g_n: \mathbb{R}^k \to \mathbb{R}$,

then if $F = f(g_1, \ldots, g_n)$ then

$$\frac{\partial F}{\partial t_i} = \frac{\partial f}{\partial x_1}\frac{\partial g_1}{\partial t_i} + \frac{\partial f}{\partial x_2}\frac{\partial g_2}{\partial t_i} + \cdots + \frac{\partial f}{\partial x_n}\frac{\partial g_n}{\partial t_i}.$$

Example $f: \mathbb{R}^4 \to \mathbb{R}$, $g, h, j, k: \mathbb{R}^3 \to \mathbb{R}$,

$$F(r,s,t) = f(g(r,s,t), h(r,s,t), j(r,s,t), k(r,s,t))$$

Then
$$\frac{\partial F}{\partial s} = \frac{\partial f}{\partial x_1}\frac{\partial g}{\partial s} + \frac{\partial f}{\partial x_2}\frac{\partial h}{\partial s} + \frac{\partial f}{\partial x_3}\frac{\partial j}{\partial s} + \frac{\partial f}{\partial x_4}\frac{\partial j}{\partial s}.$$