Lecture 11

Last time: The Chain Rule.

For $f: \mathbb{R}^3 \to \mathbb{R}$, $g, h, k : \mathbb{R}^2 \to \mathbb{R}$,

$$F(s,t) = f(g(s,t), h(s,t), k(s,t))$$

Then
$$\frac{\partial F}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial h}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial k}{\partial s}.$$

Book notation: $x(s,t)$, $y(s,t)$, $z(s,t)$.

Chain Rule: $$\frac{\partial F}{\partial s}\left(= \frac{\partial z}{\partial s}\right) = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

Example $f(x,y,z) = x^2 + yz$

$x(s,t) = s$, $y(s,t) = t$, $z(s,t) = \sqrt{1-s^2-t^2}$

What are $\frac{\partial F}{\partial s}(0,0)$, $\frac{\partial F}{\partial t}(0,0)$?

By Chain Rule:
$$\frac{\partial F}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$
$$= 2s \cdot 1 + \sqrt{1-s^2-t^2}\cdot 0 + t \cdot \frac{-s}{\sqrt{1-s^2-t^2}}$$

$$\frac{\partial F}{\partial s}(0,0) = 0$$

$$\frac{\partial F}{\partial t} = 2s \cdot 0 + \sqrt{1-s^2-t^2}\cdot 1 + t \cdot \frac{-t}{\sqrt{1-s^2-t^2}}$$

$$\frac{\partial F}{\partial t}(0,0) = 1.$$

§14.6 Directional Derivatives

Same idea as $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$.

Measure rate of change along a given direction $\vec{u}$.

If $\vec{u} = (u_1, u_2)$ then

$$D_{\vec{u}} f(\vec{c}) = \lim_{h\to 0} \frac{f(\vec{c} + h\vec{u}) - f(\vec{c})}{h}$$

$$= \lim_{h\to 0} \frac{f(c_1 + hu_1, c_2 + hu_2) - f(\vec{c})}{h}$$

<u>Note</u>: $D_{\vec{i}} f = f_x$ & $D_{\vec{j}} = f_y$.

<u>Example</u>: $f(x,y) = x - y$.

Then $f_x(x,y) = 1$, $f_y(x,y) = -1$,

$$D_{(1,1)} f(x,y) = \lim_{h\to 0} \frac{[(x+h) - (y+h)] - (x-y)}{h} = 0$$

$$D_{(1,-1)} f(x,y) = \lim_{h\to 0} \frac{[(x+h) - (y-h)] - (x-y)}{h} = \lim_{h\to 0} \frac{2h}{h} = 2$$

How to calculate in general?

Related to f_x & f_y by

<u>Theorem</u>: If f diff at $\vec{c}$, then

$$D_{\vec{u}} f(\vec{c}) = f_x(\vec{c})u_1 + f_y(\vec{c})u_2.$$

<u>Why</u>? Intersect tangent plane

$$z = f(\vec{c}) + f_x(\vec{c})(x - c_1) + f_y(\vec{c})(y - c_2)$$

with parametrized line $(c_1, c_2, f(\vec{c})) + t\,(u_1, u_2, D_{\vec{u}} f(\vec{c}))$

$$x = c_1 + tu_1, \quad y = c_2 + tu_2, \quad z = f(\vec c) + t[D_{\vec u} f(\vec c)]$$

get

$$\cancel{f(\vec c)} + t\, D_{\vec u} f(\vec c) = \cancel{f(\vec c)} + f_x(\vec c)(tu_1) + f_y(\vec c)(tu_2)$$
$$= [f_x(\vec c)u_1 + f_y(\vec c)u_2]\, t$$

So $D_{\vec u} f(\vec c) = f_x(\vec c)u_1 + f_y(\vec c)u_2$.

Formula can also be written

$$D_{\vec u} f(\vec c) = (f_x(\vec c), f_y(\vec c)) \cdot \vec u$$

Defn For $f: \mathbb{R}^2 \to \mathbb{R}$ diff., define $\nabla f: \mathbb{R}^2 \to \mathbb{R}^2$ (gradient of f) by

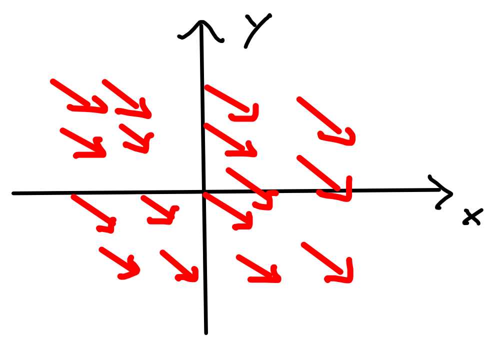
$$\nabla f(x,y) = (f_x(x,y), f_y(x,y)) = f_x \vec i + f_y \vec j.$$

Similarly, for $f: \mathbb{R}^n \to \mathbb{R}$, $\nabla f: \mathbb{R}^n \to \mathbb{R}^n$ is

$$\nabla f(\vec x) = (f_1(\vec x), f_2(\vec x), \dots, f_n(\vec x)).$$

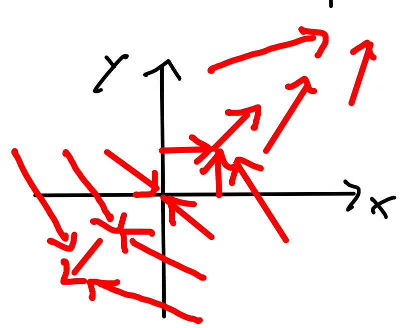
Example $f(x,y) = x - y$.

$$\nabla f(x,y) = (1, -1)$$



Example $f(x,y) = xy$

$$\nabla f(x,y) = (y, x)$$



Geometric meaning

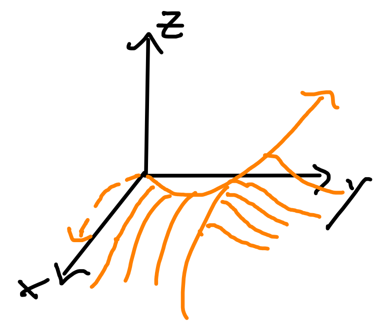
(1) $|D_{\vec{u}} f| = \|\nabla f\| \cdot \|\vec{u}\| \, |\cos\Theta|$ (usually take $\vec{u}$ a unit vector)

largest when ∇f parallel to $\vec{u}$

Take $\vec{u}$ unit vector in direction $\nabla(f)$.

Then $D_{\vec{u}} f = \|\nabla(f)\|$

Interpretation $\nabla(f)(\vec{c})$ is direction of maximal increase of f at $\vec{c}$.



Example $f(x,y) = xy$ graph is hyperbolic paraboloid

$\nabla f = (y, x)$

At $(1,1)$, $\nabla f\,(1,1) = (1,1)$.

Parabola opening up along line $y = x$.

At $(1,-1)$, $\nabla f(1,-1) = (-1,1)$

Parabola opening down along $y = -x$.

Steepest ascent back to vertex.