Lecture 12

Sept. 19
2011
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Last time: Directional derivative $D_{\vec{u}}(f)$

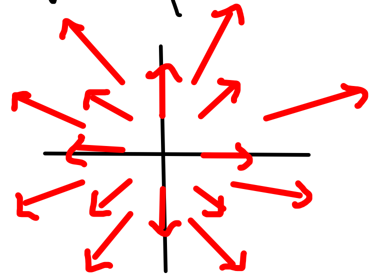
and gradient $\nabla(f) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$.

Found $D_{\vec{u}}(f) = \nabla f \cdot \vec{u}$.

Saw $\nabla(f)(\vec{c})$ = direction of maximal increase ~~of~~ of f at $\vec{c}$.

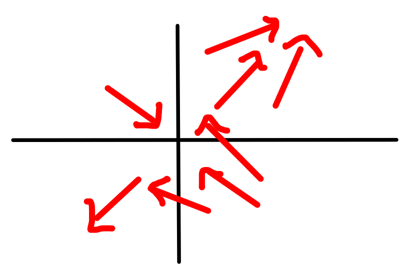
Example 1) $f(x,y) = x^2+y^2$ (elliptic paraboloid).

$\nabla(f) = (x,y) = (2x, 2y)$



2) $f(x,y) = xy$ (hyperbolic paraboloid)

$\nabla(f)(x,y) = (y,x)$



2nd geometric interpretation:

Consider level set $f(x,y) = c$.

Suppose $r(t) = (x(t), y(t))$ parametrised curve within this level set. Then (NEXT PAGE)

$$F(t) = f(x(t), y(t)) = c,$$

so $\frac{dF}{dt} = 0$.

$$\left(\frac{dx}{dt}, \frac{dy}{dt}\right)$$

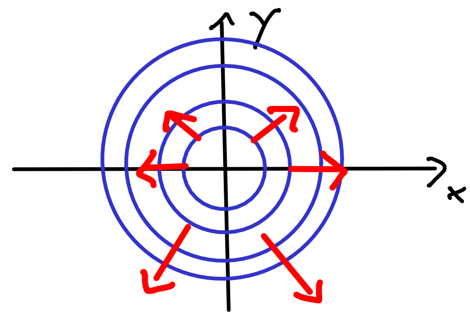
$$\text{But} \quad \frac{dF}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \nabla(f) \cdot r'(t)$$

So $\nabla(f)$ is orthogonal to $r'(t)$ (<u>tangent line</u>).

<u>Conclusion</u>: $\nabla(f)$ is orthogonal to level sets.

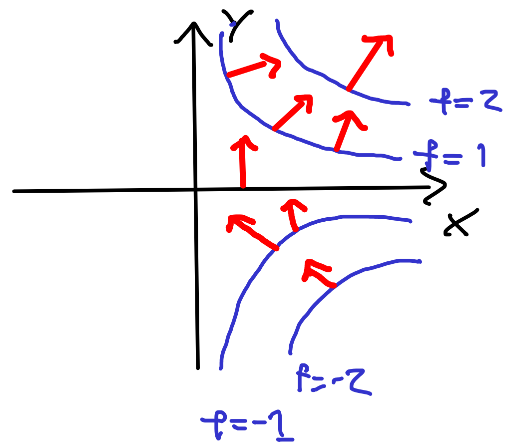
<u>Example</u> $f(x,y) = x^2 + y^2$

$$\nabla(f)(x,y) = (2x, 2y)$$



<u>Example</u> $f(x,y) = xy$

$$\nabla(f)(x,y) = (y, x).$$



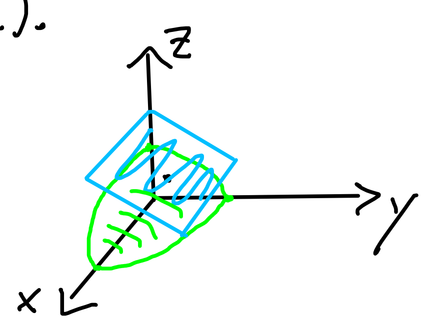
This gives another way of determining tangent planes to surfaces.

<u>Example</u>: Find tangent plane to ellipse described by $x^2 + 2y^2 + 3z^2 = 9$ at point $(2,1,1)$.

<u>Earlier approach</u> Write as graph of function

$$f(x,y) = \sqrt{\frac{9 - x^2 - 2y^2}{3}}$$

& use formula for linear approximation of f.



New (easier) approach:

Let $F(x,y,z) = x^2 + 2y^2 + 3z^2$.

Then ellipse is level surface $F = 9$.

$\nabla F = (2x, 4y, 6z)$, so $\nabla F(2,1,1) = (4,4,6)$.

$(4,4,6)$ is normal vector to tangent plane, so

plane described by $4(x-2) + 4(y-1) + 6(z-1) = 0$.

Example Find tangent plane to hyperboloid

$$z^2 + 3 = 2x^2 + 3xy + 2y^2 \quad \text{at } (1,1,2)$$

Old way Let $f(x,y) = \sqrt{2x^2 + 3xy + 2y^2 - 3}$

Then $f_x = \dfrac{4x+3y}{2\,f(x,y)}$, $f_y = \dfrac{3x+4y}{2\,f(x,y)}$.

Tangent plane described by

$$z = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1)$$
$$= 2 + \tfrac{7}{4}(x-1) + \tfrac{7}{4}(y-1).$$

New way Let $F(x,y,z) = 2x^2 + 3xy + 2y^2 - z^2$.

Then $\nabla F = (4x+3y, 3x+4y, -2z)$.

$\nabla F(1,1,2) = (7,7,-4)$.

The tangent plane is

$$7(x-1) + 7(y-1) - 4(z-2) = 0.$$