Lecture 13

Sept. 23 2011

Exam 1 info:

Average ~ 36

Letter Grade	Score	# Students
A	40-45	~90
B	34-39	~80
C	28 - 33	~60
D	21 - 27	~20
F	< 21	~10

Monday: Uses of gradient $\nabla(f)$

(1) points in direction of steepest ascent of graph f

(2) $\nabla(f)(\vec{c})$ orthogonal to level set of f at $\vec{c}$.

Other use for $\nabla(f)$:

If f has (local) maximum at $\vec{c}$, then

$\nabla(f)(\vec{c}) = \vec{0}$ (not increasing in any direction).

(If $\nabla(f)$ exists)

Example $f(x,y) = -x^2 - y^2 + 2x + 6y + 1$.

$\nabla f(x,y) = (-2x+2, -2y+6)$

$\nabla f(1,3) = \vec{0}$.

In fact, $f(x,y) = -(x-1)^2 - (y-3)^2 + 11$,
so elliptic paraboloid w/ max at $(1,3)$.

What if f has a local minimum at $\vec{c}$?

Then $-f$ has local max, so $\nabla(-f)(\vec{c}) = \vec{0}$,

$$\nabla(-f)(\vec{c}) = -\nabla(f)(\vec{c})$$

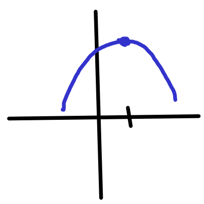
so $\nabla(f)(\vec{c}) = \vec{0}$.

Say f has critical point at $\vec{c}$ if

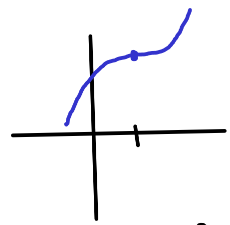
either $\nabla(f)(\vec{c}) = \vec{0}$

or $\nabla(f)(\vec{c})$ not defined.

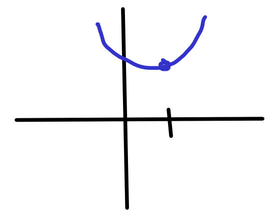
Recall for $f: \mathbb{R} \longrightarrow \mathbb{R}$, if $f'(c) = 0$, then have



max



"point of inflection"



min

Use 2nd Derivative Test:

If $f''(c) > 0$ then have min

If $f''(c) < 0$ then have max

If $f''(c) = 0$ could be any. (e.g. $x^3, x^4, -x^4$)

Now take $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$

Take slices. Max should give both

$f_{xx}(\vec{c}) < 0$ and $f_{yy}(\vec{c}) < 0$

Min gives $f_{xx}(\vec{c}) > 0$ and $f_{yy}(\vec{c}) > 0$.

But not enough.

Example $f(x,y) = x^2 - 5xy + 6y^2$

Then $f_{xx} = 2$ But this is hyperbolic

$f_{yy} = 12$ paraboloid w/ saddle point at $(0,0)$

(Make substitution $u = x-2y$, $v = x-3y$)

For $f : \mathbb{R}^2 \to \mathbb{R}$, must consider all 2^{nd} partial derivatives

Put these into 2×2 matrix (the "Hessian" of f)

$$H(f) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

Let $\overset{\text{(discriminant)}}{D} = \det H(f) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial x \partial y} \cdot \frac{\partial^2 f}{\partial y \partial x}$

$$\text{Clairaut} \longrightarrow \quad = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left[\frac{\partial^2 f}{\partial x \partial y}\right]^2$$

Examples 1) $f = x^2 + y^2$ has min at $(0,0)$.

$H(f)(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, $D = 4 > 0$

2) $f = -x^2 - y^2$ has max at $(0,0)$

$H(f)(0,0) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$, $D = 4 > 0$

3) $f = x^2 - y^2$ has saddle point at $(0,0)$

$H(f)(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$, $D = -4 < 0$

4) above example $f = x^2 - 5xy + 6y^2$

$H(f)(0,0) = \begin{pmatrix} 2 & -5 \\ -5 & 12 \end{pmatrix}$, $D = 24 - 25 = -1 < 0$

2^{nd} Derivative Test, $f: \mathbb{R}^2 \to \mathbb{R}$

Assume $\vec{c}$ is crit. point of f, 2^{nd} partials of f defined and continuous near $\vec{c}$

Then if ① $D = \det H(f)(\vec{c}) > 0$ and

a) $f_{xx}(\vec{c}) > 0$ then f has min at $\vec{c}$

or b) $f_{xx}(\vec{c}) < 0$ then f has max at $\vec{c}$

② $D = \det H(f)(\vec{c}) < 0$ then f has saddle point at $\vec{c}$.

(No information if $D = 0$)

Note: Since $D = f_{xx} f_{yy} - (f_{xy})^2$, if $D > 0$ then $f_{xx} \neq 0$.

Example: Classify critical points of

$$f(x,y) = x^3 - 12xy + 8y^3$$

$$\nabla f(x,y) = (3x^2 - 12y, -12x + 24y^2)$$

$\nabla f = \vec{0}$ when $x^2 - 4y = 0$ & $-x + 2y^2 = 0$

$x = 2y^2$

$$(2y^2)^2 - 4y = 0 \Rightarrow \boxed{y^4 = y}$$

$$(y^3 - 1) = (y-1)(y^2 + y + 1) \qquad y = 0, 1$$

$x = 0$ $\quad$ $x = 2$

crit points $(0,0)$ and $(2,1)$

$$H(f) = \begin{pmatrix} 6x & -12 \\ -12 & 48y \end{pmatrix}, \quad D = 288xy - 144$$
$$= 144(2xy - 1).$$

$D(0,0) = -144 < 0$ saddle point

$D(2,1) = 144 \cdot 3 = 432 > 0$

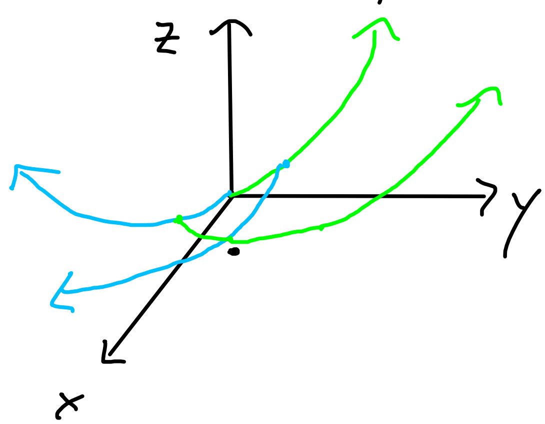
Check $f_{xx} = 6x = 12 > 0$ at $(2,1)$. So $(2,1)$ is (local) min.

$f(2,1) = 8 - 12 \cdot 2 + 8 \cdot 1 = -8.$

Is $(2,1)$ a global max? No. $f(-10,0) = -1000$.

f is not "bounded below".

Point $(2,1)$ is only minimum near $(2,1)$.



$f(x,1) = x^3 - 12x + 8$

has local min at 2

$f(2,y) = 8 - 24y + 8y^3$