Lecture 14

Sept. 26
2011
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Last time: 2^{nd} Deriv. Test for $f: \mathbb{R}^2 \to \mathbb{R}$.

discriminant $D = f_{xx} f_{yy} - [f_{xy}]^2$.

If $\nabla(f)(\vec{c}) = \vec{0}$ &

$D > 0$ then $\begin{cases} f_{xx} > 0 & \rightsquigarrow f \text{ has min at } \vec{c} \\ f_{xx} < 0 & \rightsquigarrow f \text{ has max at } \vec{c} \end{cases}$

$D < 0$ then f has saddle point at $\vec{c}$.

Note When $D > 0$, rule says to look at f_{xx}, not f_{yy}. Get same answer if use f_{yy} instead. Why?

If $D > 0$, then $f_{xx} f_{yy} > (f_{xy})^2$, so $f_{xx} f_{yy} > 0$.

So $f_{xx} > 0 \longleftrightarrow f_{yy} > 0$,

$f_{xx} < 0 \longleftrightarrow f_{yy} < 0$.

For $f: \mathbb{R}^n \to \mathbb{R}$, $n > 2$, similar test, but need to consider "eigenvalues" of Hessian matrix $H(f)$.

2^{nd} Deriv. Test useful for minimization problems

Example Find distance between point $Q = (-1, 3, 0)$ and plane P given by $-2x + y + z = 3$.

Did this on Aug. 30 Worksheet using different method.

Here, want to minimize distance function

$$d(x,y,z) = \sqrt{(x+1)^2 + (y-3)^2 + z^2}$$

subject to constraint $-2x + y + z = 3$.

Easier (but equivalent) to minimize $d^2 = (x+1)^2 + (y-3)^2 + z^2$.

Substitute $z = 2x - y + 3$ into d^2, get

$$f(x,y) = (x+1)^2 + (y-3)^2 + (2x-y+3)^2$$

$$= x^2 + 2x + 1 + y^2 - 6y + 9 + 4x^2 - 4xy + y^2 + 12x - 6y + 9$$

$$= 5x^2 - 4xy + 2y^2 + 14x - 12y + 19$$

Find minimum of f.

$$\nabla(f) = (10x - 4y + 14,\ -4x + 4y - 12)$$

So $\nabla(f) = 0$ when

$$10x - 4y + 14 = 0 \quad \& \quad -4x + 4y - 12 = 0$$

substitute

$$x = y - 3$$

get $10(y-3) - 4y + 14 = 0$

or $\quad 6y = 16 \longrightarrow y = 8/3,\ x = -1/3.$

Know f can't have max here (from the geometry).

2nd Der Test:

$$f_{xx} = 10, \quad f_{xy} = -4, \quad f_{yy}\ 4$$

$$D = 10 \cdot 4 - (-4)^2 = 40 - 16 = 24 > 0$$

and $f_{xx} = 10 > 0$,

so every crit point of f is a (local) min.

The point $(-1/3, 8/3, -1/3)$ on $\mathcal{P}$ minimizes the distance, and $d = \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{-1}{3}\right)^2 + \left(\frac{-1}{3}\right)^2}$

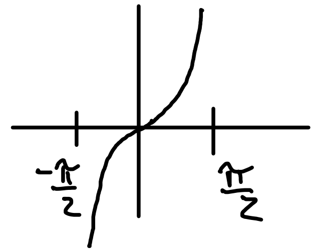
$$= \sqrt{6/9} = \frac{\sqrt{6}}{3}.$$

In example found (global) min but no global max.
When does global min/max exist?

For $f: \mathbb{R} \to \mathbb{R}$, have "closed interval".

<u>Extreme Value Theorem</u> If f is continuous on $[a,b]$ then f has max & min on $[a,b]$.

Not true if replace $[a,b]$ with (a,b):



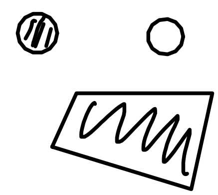
$\tan x$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$ has no max or min.

<u>Key</u> $[a,b]$ is <u>closed</u> (includes endpoints) and <u>bounded</u> (contained within <u>some</u> interval $[-r,r]$).

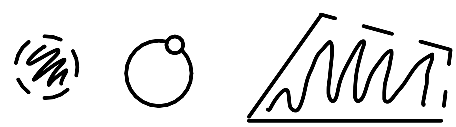
For Ω (Omega) region in $\mathbb{R}^2$, say Ω is <u>closed</u> if contains all boundary points.

Say Ω is <u>bounded</u> if contained in some disk $\{(x,y) \mid \|(x,y)\| \le r\}$.

<u>Closed</u>



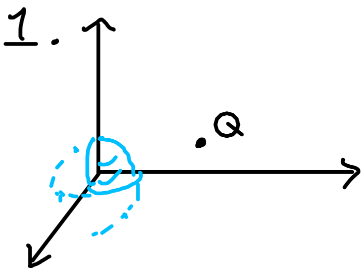
<u>Not closed</u>



<u>Extreme Value Theorem</u> ($f: \mathbb{R}^2 \to \mathbb{R}$)
If Ω is closed and bounded in $\mathbb{R}^2$ and f is continuous on Ω, then f has global max & min on Ω.

<u>Example</u> $f =$ distance squared to $Q = (-1, 3, 0)$ restricted to $\Omega =$ sphere at origin of radius 1.
Restrict to northern hemisphere, so

$$(x,y,z) = (x, y, \sqrt{1-x^2-y^2}).$$



Then
$$f(x,y) = (x+1)^2 + (y-3)^2 + 1 - x^2 - y^2$$
$$= 2x - 6y + 11.$$

Look for critical points:

$\nabla f = (2, -6)$ never $\vec{0}$, so no critical points.

Also check the boundary: $z=0$

Better to switch to polar coordinates. $x = \cos\theta$, $y = \sin\theta$,

$F(\theta) = f(\cos\theta, \sin\theta) = 2\cos\theta - 6\sin\theta + 11.$

$F'(\theta) = -2\sin\theta - 6\cos\theta$

$F'(\theta) = 0$ at $\theta = \arctan(-3) \approx -72°, 108°$

$\left(x(\arctan(-3)), y(\arctan(-3)) \right) = \left(\frac{1}{\sqrt{10}}, \frac{-3}{\sqrt{10}} \right)$

or $\left(\frac{-1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \right)$

Use $\cos(\arctan x) = \frac{1}{\sqrt{1+x^2}}$
$\sin(\arctan x) = \frac{x}{\sqrt{1+x^2}}$

$F''(\theta) = -2\cos\theta + 6\sin\theta$

$F''(\arctan -3) = \frac{-2-18}{\sqrt{10}} < 0$ at $\left(\frac{1}{\sqrt{10}}, \frac{-3}{\sqrt{10}} \right)$ so max

$F'' = \frac{2+18}{\sqrt{10}} > 0$ at $\left(\frac{-1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \right)$ so min.

$f\left(\frac{1}{\sqrt{10}}, \frac{-3}{\sqrt{10}} \right) = \frac{2+18}{\sqrt{10}} + 11 = 2\sqrt{10} + 11 \approx 17.3$

$f\left(\frac{-1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \right) = \frac{-2-18}{\sqrt{10}} + 11 = -2\sqrt{10} + 11 \approx 4.7$