Lecture 15

Sept. 28 2011

Last time: Global maximization / minimization

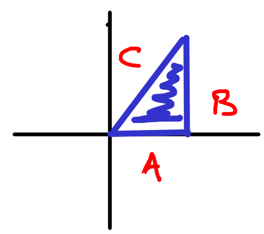
General strategy for find min/max of continuous f on closed & bounded region Ω:

1. Find crit points & use 2^{nd} Deriv Test
2. Find extreme values on boundary of Ω

Then compare findings to get global min/max.

Ex Find min & max of $f(x,y) = 9x^2 + 3xy - 6y^2 - 9x + 11y$

on (solid) triangle Ω w/ vertices $(0,0)$, $(1,0)$, $(1,2)$



$$\nabla f = (18x + 3y - 9,\ 3x - 12y + 11)$$

$\nabla f = \vec{0}$ when $6x + y - 3 = 0$ & $3x - 12y + 11 = 0$

$y = 3 - 6x$, $\quad 3x - 12(3 - 6x) + 11 = 0$

$75x - 25 = 0 \rightsquigarrow x = \frac{1}{3}$

$y = 3 - 6(\frac{1}{3}) = 1$. $\quad (\frac{1}{3}, 1)$ Outside of triangle!

No critical points on Ω.

Check boundary too. 3 Pieces.

A) $y = 0$, $0 \le x \le 1$ $\quad f(x,0) = 9x^2 - 9x = 9(x - \frac{1}{2})^2 - \frac{9}{4}$

has min at $(\frac{1}{2}, 0)$ $\quad f(\frac{1}{2}, 0) = -\frac{9}{4}$

max at $(0,0)$ & $(1,0)$ $\quad f(0,0) = f(1,0) = 0$.

B) $x = 1$, $0 \le y \le 2$ $\quad f(1,y) = -6y^2 + 14y = -6(y^2 - \frac{7}{3}y)$

$= -6(y - \frac{7}{6})^2 + \frac{49}{6}$

has max at $(1, \frac{7}{6})$ $\quad f(1, \frac{7}{6}) = \frac{49}{6} \approx 8.2$

min at $(1,0)$ $\quad f(1,0) = 0$

c) $y=2x$, $0 \le x \le 1$ $\quad f(x,2x) = 9x^2+6x^2-24x^2-9x+22x$

$$= -9x^2 + 13x = -9\left(x^2 - \tfrac{13}{9}x\right)$$

$$= -9\left(x - \tfrac{13}{18}\right)^2 + \frac{(13)^2}{36} \approx 4.7$$

has max at $\left(\frac{13}{18}, \frac{26}{18}\right)$

$f\left(\frac{13}{18}, \frac{26}{18}\right) \approx 4.7$

min at 0, $f(0,0) = 0$.

Global min on Ω is $f(\frac{1}{2}, 0) = -4.25$

global max on Ω is $f(1, 7/6) \approx 8.17$

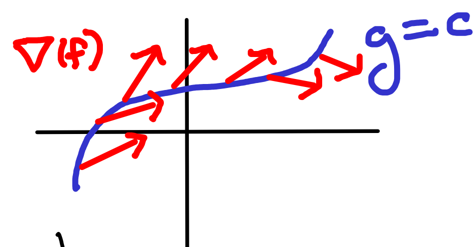
---

§14.8 Lagrange Multipliers

Another maximization technique:

$f: \mathbb{R}^2 \to \mathbb{R}$. Maximize f on level curve of another function $g: \mathbb{R}^2 \to \mathbb{R}$.

Consider $\nabla(f)$ on level curve $g=c$.



$\nabla(f)$ points in direction of steepest ascent, so unless $\nabla(f) \perp$ level curve, can increase f by moving along level curve.

Conclusion Maximum can only happen where $\nabla(f)$ is orthogonal to level set of g.

But recall $\nabla(g)$ always orthog. to level curve of g.

So Max (or min) can only happen where

$$\nabla(f)(\vec{x}) = \lambda \nabla(g)(\vec{x})$$

any constant ("Lagrange multiplier")

Example: Maximize function

$f(x,y) = x^2 y + y^2$ on unit circle

Constraint is level curve $\underbrace{x^2+y^2}_{g(x,y)} = 1$

Can we find x, y, λ so that

$$\underset{\displaystyle (2xy,\ x^2+2y)}{\underset{\shortparallel}{\nabla f(x,y)}} = \underset{\displaystyle \lambda(2x, 2y)}{\underset{\shortparallel}{\lambda \nabla g(x,y)}} \quad \text{and} \quad x^2+y^2=1 \ ?$$

$\underbrace{2xy = 2x\lambda}$, $x^2+2y = 2y\lambda$, $x^2+y^2=1$.

either $x=0$ or $y=\lambda$

$\underline{x=0}$: $y=\pm 1$, $\lambda = 1$

1, 2 $f(0,1) = f(0,-1) = 1$

$\underline{y=\lambda}$: $x^2+2y = 2y^2$, $x^2 = 1-y^2$

$\Rightarrow 1-y^2+2y = 2y^2$ or $3y^2-2y-1=0$

$(3y+1)(y-1)$

$y = 1, -\frac{1}{3}$

$y=1$, $\lambda = 1$, $x=0$. Already found.

3, 4 $y = -\frac{1}{3}$, $\lambda = -\frac{1}{3}$, $x = \pm\frac{2\sqrt{2}}{3}$

$f\left(\pm\frac{2\sqrt{2}}{3}, -\frac{1}{3}\right) = \frac{8}{9}\cdot\left(-\frac{1}{3}\right) + \frac{1}{9} = \frac{-5}{27}$.

Max value of 1 at $(0,1)$ & $(0,-1)$.

Min value of $\frac{-5}{27}$ at $\left(\frac{2\sqrt{2}}{3}, -\frac{1}{3}\right), \left(\frac{-2\sqrt{2}}{3}, -\frac{1}{3}\right)$

Works same way for functions of 3 vars (or more).

Maximize $f(x,y,z)$ on level set $g(x,y,z) = k$.
(surface)

If $\nabla(f)$ not orthog. to surface, may increase f by moving along surface. $\nabla(g)$ orthog. to surface, so

$\nabla(f) = \lambda \nabla(g)$ at extreme values.

Example (From Monday) f = distance squared to $Q = (-1,3,0)$ on unit sphere.

$$f(x,y,z) = (x+1)^2 + (y-3)^2 + z^2, \quad 1 = g(x,y,z) = x^2+y^2+z^2$$

Solve $\nabla(f) = \lambda \nabla(g)$

$$(2x+2,\ 2y-6,\ 2z) = \lambda(2x, 2y, 2z)$$

$$2x+2 = 2x\lambda, \quad 2y-6 = 2y\lambda, \quad \underbrace{2z = 2z\lambda}_{z=0 \text{ or } \lambda = 1.}$$

No solutions to if $\lambda = 1$.

So $z=0$, $x^2+y^2 = 1$

$2x+2 = 2x\lambda \rightsquigarrow \quad x = \frac{1}{\lambda-1}$

$2y-6 = 2y\lambda \rightsquigarrow \quad y = \frac{-3}{\lambda-1}$.

$$1 = \left(\frac{1}{\lambda-1}\right)^2 + \left(\frac{-3}{\lambda-1}\right)^2 = \frac{10}{(\lambda-1)^2}, \text{ so } (\lambda-1)^2 = 10$$

$$\lambda^2 - 2\lambda - 9 = 0, \text{ so } \lambda = 1 \pm \sqrt{10}.$$

$\underline{\lambda = 1+\sqrt{10}}$ $\quad x = \frac{1}{-\sqrt{10}}, \quad y = \frac{-3}{-\sqrt{10}} = \frac{3}{\sqrt{10}}$

$$f\left(-\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}, 0\right) = -2\sqrt{10} + 11 \approx 4.7$$

$\lambda = 1-\sqrt{10}$ $\quad x = \frac{1}{\sqrt{10}}, \quad y = \frac{-3}{\sqrt{10}}$

$$f\left(\frac{1}{\sqrt{10}}, \frac{-3}{\sqrt{10}}, 0\right) = 2\sqrt{10} + 11 \approx 17.3$$