Lecture 16

Sept. 30
2011
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Quiz on Tuesday (14.7, 14.8)

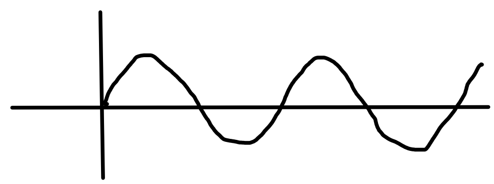
Ch. 13 Vector (-valued) Functions

Already seen vector function $\nabla f: \mathbb{R}^3 \to \mathbb{R}^3$.
Studied $f: \mathbb{R}^3 \to \mathbb{R}$, $f: \mathbb{R}^2 \to \mathbb{R}$.

Now: $f: \mathbb{R} \to \mathbb{R}^2$, $f: \mathbb{R} \to \mathbb{R}^3$.

Ex $\vec{r}(t) = (t, \sin t)$.

parametrizes graph of $\sin(t)$

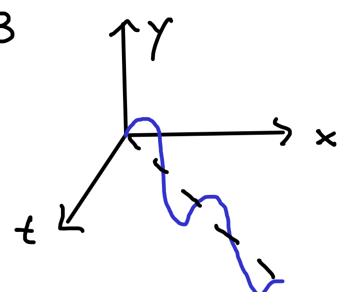


write $\vec{r}(t) = (x(t), y(t))$

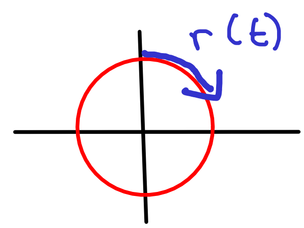
$x(t)$, $y(t)$ "components" of $\vec{r}(t)$

May also write $x(t) = r_1(t)$, $y(t) = r_2(t)$ (conflicts w/ partial deriv notation)

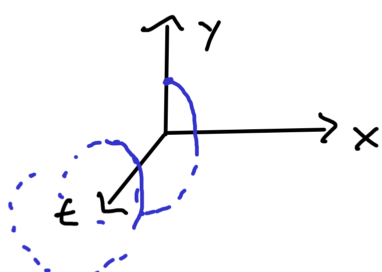
graph of $\vec{r}(t)$ in $\mathbb{R}^3$



Ex $\vec{r}(t) = (\sin t, \cos t)$ parametrizes unit circle (starting at $(0,1)$, clockwise)



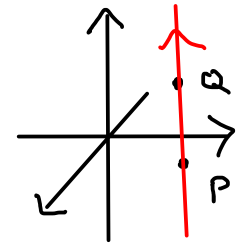
graph of $\vec{r}(t)$:



Ex $\vec{r}(t)$ parametrization of line through $P = (1, 3, 0)$ & $Q = (-2, 2, 1)$

at time 0, $\vec{r}(0) = P = (1, 3, 0)$.

time 1, $\vec{r}(1) = Q = P + \overrightarrow{PQ}$



time t, $\vec{r}(t) = P + t\,\overrightarrow{PQ}$

$$= (1,3,0) + t\,(-3,-1,1)$$

$$= (\underbrace{1-3t}_{x(t)}, \underbrace{3-t}_{y(t)}, \underbrace{t}_{z(t)})$$

__Example__ Give parametrization of curve where cylinder $x^2+y^2=1$ meets plane $2y+3z=1$. Since $z = \frac{1-2y}{3}$, this is

$$\vec{r}(t) = \left(\cos t, \sin t, \frac{1-2\sin t}{3}\right).$$

__Limits?__ $\vec{r}(t) = (x(t), y(t))$.

__Roughly__ $\lim_{t\to c} \vec{r}(t) = \vec{u}$ if $\underbrace{\vec{r}(t) \text{ near } \vec{u}}_{\|\vec{r}(t)-\vec{u}\| \text{ small}}$ for $\underbrace{t \text{ near } c}_{|t-c| \text{ small}}$.

$$\sqrt{(x(t)-u_1)^2 + (y(t)-u_2)^2}$$

But $\|\vec{r}(t)-\vec{u}\|$ small $\longleftrightarrow$ $|x(t)-u_1|$ small & $|y(t)-u_2|$ small.

So $\lim_{t\to c} \vec{r}(t) = \vec{u} \longleftrightarrow \lim_{t\to c} x(t) = u_1$ & $\lim_{t\to c} y(t) = u_2$.

So $\lim_{t\to c} \vec{r}(t) = \left(\lim_{t\to c} x(t), \lim_{t\to c} y(t)\right)$

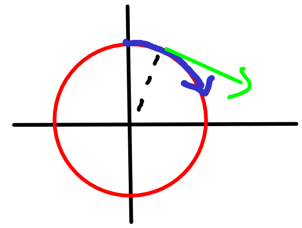
Derivatives?

$$\frac{d\vec{r}}{dt}(c) = \lim_{t\to c} \frac{\vec{r}(t)-\vec{r}(c)}{t-c} = \lim_{t\to c} \frac{(x(t),y(t)) - (x(c),y(c))}{t-c}$$

$$= \lim_{t\to c} \left(\frac{x(t)-x(c)}{t-c}, \frac{y(t)-y(c)}{t-c}\right)$$

$$= \left(\lim_{t\to c} \frac{x(t)-x(c)}{t-c}, \lim_{t\to c} \frac{y(t)-y(c)}{t-c}\right) = (x'(c), y'(c))$$

$\vec{r}\,'(c)$ is tangent vector to curve parametrized by $\vec{r}(t)$ at $t=c$.

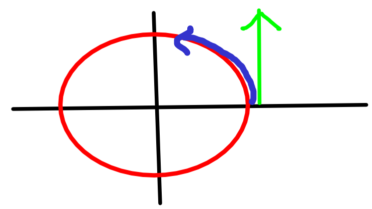


Ex $\vec{r}(t) = (\sin t, \cos t)$.

$\vec{r}\,'(t) = (\cos t, -\sin t)$.

$\vec{r}\,'(\frac{\pi}{6}) = (\frac{\sqrt{3}}{2}, \frac{-1}{2})$

Parametrized tangent line: $(\frac{1}{2}, \frac{\sqrt{3}}{2}) + t(\frac{\sqrt{3}}{2}, -\frac{1}{2}) = (\frac{1+t\sqrt{3}}{2}, \frac{\sqrt{3}-t}{2})$



Ex $\vec{r}(t) = (3\cos t, 2\sin t)$.

parametrizes ellipse

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

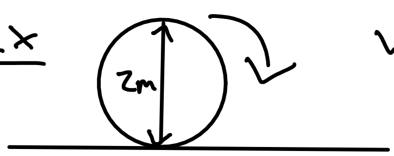
$\vec{r}\,'(t) = (-3\sin t, 2\cos t)$ $\quad \vec{r}\,'(0) = (0,2)$

Parametrized tangent line: $(3,0) + t(0,2) = (3, 2t)$.

Physical interpretation: $\vec{r}(t)$ = position of particle at time t.

$\vec{r}\,'(t)$ = velocity vector at time t, $\|\vec{r}\,'(t)\|$ = speed at time t.

Ex



Wheel moving forward at constant angular speed of 1 radian/sec. Find velocity, speed, acceleration of particle stuck to top of wheel.

wheel does one revolution in 2π seconds, moves 2π meters.

So wheel moving at 1 m/sec.

$\vec{r}(t) = (t, 1) + (\sin t, \cos t) = (t + \sin t, 1 + \cos t)$.

$\vec{r}\,'(t) = (1 + \cos t, -\sin t)$ velocity

$\|\vec{r}\,'(t)\| = \sqrt{(1+\cos t)^2 + (-\sin t)^2} = \sqrt{1 + 2\cos t + \cos^2 t + \sin^2 t}$

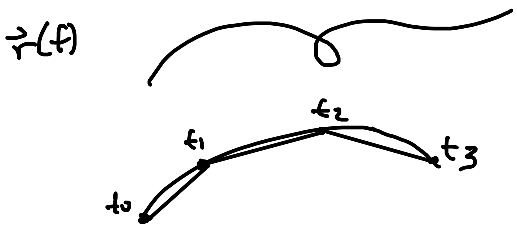
speed $= \sqrt{2 + 2\cos t}$

$\vec{r}\,''(t) = (-\sin t, -\cos t)$ acceleration

centripetal acceleration.

Arc Length

How to measure length of a curve?



Approximate by straight lines.

Measure $\|\vec{r}(t_1)-\vec{r}(t_0)\| + \|\vec{r}(t_2) - \vec{r}(t_1)\|$
$+ \|\vec{r}(t_3)-\vec{r}(t_2)\| \cdots$.

Use Linear Approximation: $\vec{r}(t) \approx \vec{r}(t_0) + \vec{r}\,'(t_0)(t_1 - t_0)$.

So $\|\vec{r}(t_1)-\vec{r}(t_0)\| \approx \|\vec{r}\,'(t_0)\|(t_1-t_0)$

Arc length $\approx \|\vec{r}\,'(t_0)\|(t_1-t_0) + \|\vec{r}\,'(t_1)\|(t_2-t_1)$
$+ \|\vec{r}\,'(t_2)\|(t_3-t_2) + \cdots$

Looks like Riemann Sum

$$\boxed{\text{Arc Length} = \int \|\vec{r}\,'(t)\|\,dt}$$

For curve $y = f(x)$, may have seen formula

$$\text{Arc length} = \int \sqrt{1+[f'(x)]^2}\,dx$$

How do these match up? Assume $y = f(x)$. Then

$y(t) = f(x(t))$, so $y'(t) = f'(x)\cdot x'(t)$ (Chain Rule).

Then $\|\vec{r}\,'(t)\| = \sqrt{[x'(t)]^2 + [y'(t)]^2} = \sqrt{[x'(t)]^2 + [f'(x)]^2[x'(t)]^2}$

$= x'(t)\sqrt{1+[f'(x)]^2}$

So $\int \|\vec{r}\,'(t)\|\,dt = \int \sqrt{1+[f'(x)]^2}\;x'(t)\,dt$

$= \int \sqrt{1+[f'(x)]^2}\,dx$.