Lecture 17

Oct. 3 2011

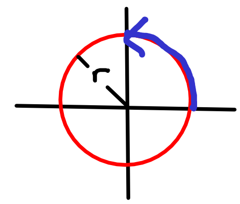
Quiz on Tuesday (14.7, 14.8)

Last time: Curves in $\mathbb{R}^2$, $\mathbb{R}^3$.

Arc Length: Length of curve $\vec{r}(t)$ from time t_0 to t_1

is $$\text{Arc Length} = \int_{t_0}^{t_1} \|\vec{r}\,'(t)\|\, dt$$

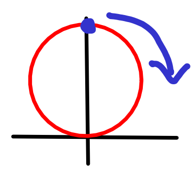
Ex $\vec{p}(t) = (r\cos t, r\sin t)$ $\quad 0 \le t \le 2\pi$



circumference of circle of radius r $$= \int_0^{2\pi} \sqrt{(-r\sin t)^2 + (r\cos t)^2}\, dt$$

$$= \int_0^{2\pi} r\, dt = 2\pi r$$

Ex from last time

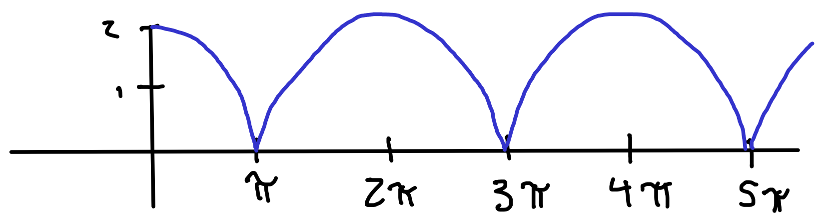


1 radian/sec

$$\vec{r}(t) = (\sin t, \cos t) + (t, 1)$$
$$= (\sin t + t, \cos t + 1)$$

Curve $\vec{r}(t)$

"cycloid"



What is arc length for $0 \le t \le \pi$?

Already calculated $\|r'(t)\| = \sqrt{2 + 2\cos t}$
(speed)

So $$\text{Arc Length} = \sqrt{2}\int_0^{\pi} \sqrt{1+\cos t}\, dt$$

Use half angle formula:
$$\cos t = 2\cos^2 t/2 - 1$$

$$= \sqrt{2}\int_0^{\pi} \sqrt{2\cos^2 t/2}\, dt$$

$$= 2\int_0^{\pi} \cos t/2\, dt = 4\int_0^{\pi/2} \cos u\, du = 4$$

Careful!

$\vec{r}(t)$ gives parametrization of a curve, but arc length should not depend on choice of parametrization. ②

<u>Ex</u> Above, saw circumference $= 2\pi r$. So arclength of semicircle, parametrized by $(r\cos t, r\sin t)$ for $0 \le t \le \pi$, is πr.

Can also parametrize by $(-x, \sqrt{r^2-x^2})$ $-r \le x \le r$.

Then arc length is

$$\int_{-r}^{r} \left\| \left(-1, \frac{-x}{\sqrt{r^2-x^2}}\right) \right\| dx = \int_{-r}^{r} \sqrt{1 + \frac{x^2}{r^2-x^2}}\, dx$$

$$= \int_{-r}^{r} \sqrt{\frac{r^2}{r^2-x^2}}\, dx = r \int_{-r}^{r} \frac{1}{\sqrt{r^2-x^2}}\, dx$$

$x = r\sin\theta$

$dx = r\cos\theta\, d\theta$ $\quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$$= r \int_{-\pi/2}^{\pi/2} \frac{1}{\cancel{r}\sqrt{1-\sin^2\theta}} \cancel{r}\cos\theta\, d\theta$$

$$= r \int_{-\frac{\pi}{2}}^{\pi/2} \frac{\cos\theta}{\cos\theta}\, d\theta = r \cdot \left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = r\pi.$$

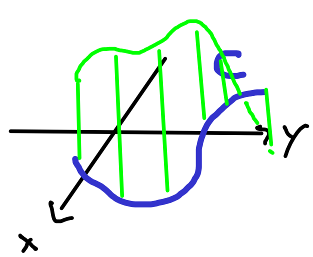
careful!

Same answer, different parametrization.

---

§16.2 Integration along a curve ("Line integrals")

$f: \mathbb{R}^2 \to \mathbb{R}$ $\qquad$ C curve in $\mathbb{R}^2$, parametrized by $\vec{r}(t)$.



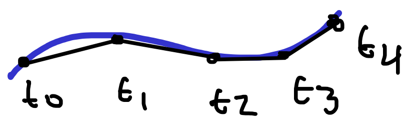
<u>Ex</u> $f(x,y)$ = elevation function.
C = hiking trail.

$\vec{r}(t) = x(t), y(t)$ = position (along trail) at time t.

Then $\frac{1}{t_1 - t_0} \int_{t_0}^{t_1} f(x(t), y(t))\, dt$ measures hiker's average elevation, with respect to time. ③

What is actual average elevation on trail?

Idea



(Like arc length)

approximate by straight lines, measure elevation at an endpoint. Use length · height (elevation)

Recall $\|\vec{r}(t_1) - \vec{r}(t_0)\| \approx \|\vec{r}\,'(t_0)\|(t_1 - t_0)$

So approximate by

$$\frac{1}{\text{length}(C)} \underbrace{\sum_{i=0} \|\vec{r}\,'(t_i)\|(t_{i+1} - t_i) \cdot f(x(t_i), y(t_i))}_{\text{Riemann sum}}$$

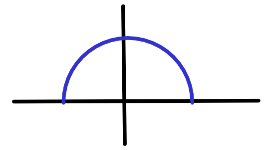
Average should be $\displaystyle \frac{1}{\text{length}(C)} \int_{t_0}^{t_{end}} \|\vec{r}\,'(t)\| \, f\circ\vec{r}(t) \, dt.$

Write $\displaystyle \int_C f(x,y)\,ds = \int_{t_0}^{t_{end}} \|\vec{r}\,'(t)\| \, f(\vec{r}(t))\,dt$

s ← arc length parameter.

The "line integral" of f along C. Does not depend on choice of $\vec{r}(t)$ (make sure parametrization only traverses path once)

Ex Find $\displaystyle \int_C x^2 y \, ds$, where s is upper unit semi-circle



Use either Ⓐ $\vec{r}(\theta) = (\cos\theta, \sin\theta)$ or Ⓑ $\vec{r}(t) = (-t, \sqrt{1-t^2})$

Ⓐ $\displaystyle \int_C x^2 y \, ds = \int_0^{\pi} \sqrt{(-\sin\theta)^2 + \cos^2\theta} \; \cos^2\theta \sin\theta \, d\theta$

$u = \cos\theta$

$du = -\sin\theta \, d\theta$

$\displaystyle = \int_0^{\pi} \cos^2\theta \sin\theta \, d\theta$

$\displaystyle = -\int_1^{-1} u^2 \, du = -\frac{u^3}{3}\Big|_1^{-1} = 2/3.$

Using (B) $\int_C x^2 y\, ds = \int_{-1}^{1} \sqrt{(-1)^2 + \frac{(-t)^2}{1-t^2}}\, (-t)^2 \sqrt{1-t^2}\, dt$

$$= \int_{-1}^{1} \frac{1}{\cancel{\sqrt{1-t^2}}}\, t^2 \cancel{\sqrt{1-t^2}}\, dt$$

$$= \int_{-1}^{1} t^2\, dt = \frac{t^3}{3}\Big|_{-1}^{1} = 2/3 .$$

Average value $= \frac{1}{\text{length}} \int_C f\, ds = \frac{1}{\pi} \cdot 2/3 = \frac{2}{3\pi}$