Lecture 18

Oct. 5 2011

<u>Correction</u> Average value of function f along curve C $= \dfrac{1}{\text{length } C} \int_C f\,ds$

Computed $\int_C x^2 y\,ds = 2/3$, C upper unit semicircle.

So average value $= \dfrac{1}{\pi} \cdot 2/3 = \dfrac{2}{3\pi}$

Recall $\int_C f\,ds = \int_{t_0}^{t_1} f(\vec{r}(t))\,\|\vec{r}\,'(t)\|\,dt$

if $\vec{r}(t)$ parametrices curve C.

<u>Alternative interpretation</u> Sometimes, can write $\vec{r}(t)$ as function of s. Then ordinary interpretation of $\int f \circ \vec{r}\,ds$ works.

<u>Ex</u> Above $\vec{r}(\theta) = (\cos\theta, \sin\theta)$. θ already <u>is</u> arclength from $(1,0)$ to point $\vec{r}(\theta)$. So already function of arclength, $ds = d\theta$.

<u>Ex</u> (Simple). Find average value of $f(x,y) = x^2 y$ on segment given by $\vec{r}(t) = (4t, 3t)$ for $0 \le t \le 1$.

Here $s = 5t$, so $t = s/5$. $\vec{r}(t(s)) = (4/5\, s, 3/5\, s)$.

$$\int_0^5 (4/5\, s)^2 (3/5\, s)\,ds = \frac{48}{125} s^4/4 \Big|_0^5 = \frac{12}{125} \cdot 125 \cdot 5$$

$$= 60.$$

Average $= \frac{1}{5} \cdot 60 = \underline{12}$.

<u>Ex</u> Find average value of $f(x,y,z) = x + z$ on C given by $\vec{r}(t) = (t, 4/3\, t^{3/2}, t^2)$, $0 \le t \le 2$

First, find length(C)

$\vec{r}\,'(t) = (1,\ 2\sqrt{t} + 2t)$

$\|\vec{r}\,'(t)\| = \sqrt{1 + 4t + 4t^2}$

$= 1 + 2t$

So $\text{length}(C) = \int_0^2 \|\vec{r}\,'(t)\|\, dt$

$= \int_0^2 1 + 2t\, dt = t + t^2 \Big|_0^2 = 6$

This tells us $s(t) = t + t^2$ (arc length as function of time)

Can solve for t: $\sqrt{s + 1/4} - 1/2 = t$

Then average value of f on C is

$$\frac{1}{6} \int_C f\, ds = \frac{1}{6} \int_C x(t) + z(t)\, ds$$

length(C)

$$= \frac{1}{6} \int_C t + t^2\, ds$$

$$= \frac{1}{6} \int_0^6 s\, ds = \frac{s^2}{12}\Big|_0^6 = 3$$

Note: Since average of $f(x,y) = 1$ is 1,

then $1 = \frac{1}{\text{length}(C)} \int_C ds$ so $\int_C ds = \text{length } C.$

$ds = \|\vec{r}\,'(t)\|\, dt$ for any parametrization $\vec{r}(t)$.

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§ 16.1 Vector Fields

Just a function $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$

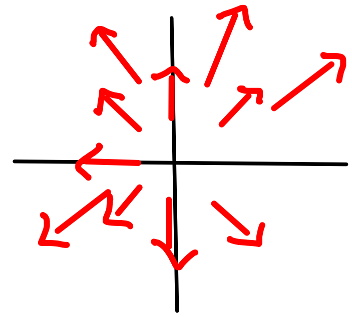
or $\mathbb{R}^3 \longrightarrow \mathbb{R}^3$

Ex gradient vector field

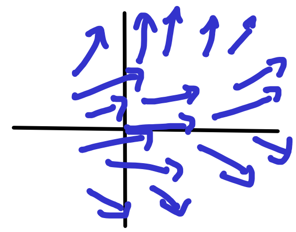
$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$f(x,y) = x^2 + y^2$

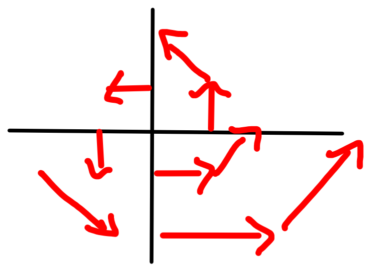
$\nabla f = (2x, 2y)$



Ex Wind velocity vector field



Ex $\vec{F}(x,y) = -y\vec{i} + x\vec{j}$



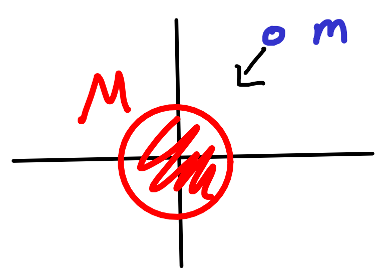
Note $\vec{F}(x,y) \cdot (x,y)$

$= -yx + xy = 0,$

so $\vec{F}(x,y) \perp (x,y)$. **Conclusion:** $\vec{F}(x,y)$ tangent to circle around $(0,0)$ through (x,y).

Ex Gravitational field (Force exerted by large mass M on small mass m).

Force $\vec{F}$ depends on position of m.



Newton's Law

$$\|\vec{F}\| = \frac{mMG}{\|\vec{r}\|^2}$$

G: gravitational constant

$\vec{r}$: position vector for m. See that $\vec{F}(\vec{r}) = c(-\vec{r})$, some $c > 0$. What is c?

$\|\vec{F}\| = c\|\vec{r}\|$ and $\|\vec{F}\| = \dfrac{mMG}{\|\vec{r}\|^2}$. So $c = \dfrac{mMG}{\|\vec{r}\|^3}$.

$$\vec{F}(\vec{r}) = -\frac{mMG}{\|\vec{r}\|^3}\vec{r}$$

In fact, if $f(x,y) = \dfrac{mMG}{\|\vec{r}\|}$, then

$$\nabla(f) = \left(-\frac{mMG}{\cancel{2}\|\vec{r}\|^3}\cancel{2}x, -\frac{mMG}{\cancel{2}\|\vec{r}\|^3}\cancel{2}y\right)$$

$$= \frac{-mMG}{\|\vec{r}\|^3}\vec{r}$$

so $\vec{F} = \nabla(f)$.

Say $\vec{F}$ is a <u>conservative</u> vector field if it is $\nabla(f)$ for some function f. So $\vec{F}$ is conservative.

Is <u>every</u> vector field conservative? NO.

Will see $\vec{F}(x,y) = (-y, x)$ is <u>not</u> conservative.

(Hint: $\vec{F}$ tangent to circle, so if $\vec{F} = \nabla f$, then f should always increase if travel once around circle. Impossible!)

If $\vec{F} = \nabla(f)$, call f <u>potential function</u> for $\vec{F}$.

(Think of f as function of "potential energy".)