Lecture 19

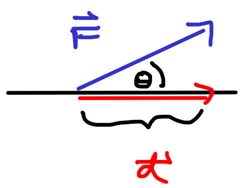
Oct. 7 2011

Last time: • Line integrals of functions
• Vector fields $\vec{F}$

Today: Line integrals of vector fields $\int_C \vec{F}$ & Fundamental Theorem for Line Integrals

Integrating vector fields.

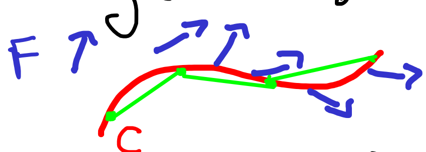
First, recall Work = force · distance.



Component of force in direction $\vec{d}$ is $\|\vec{F}\|\cos\theta$, so

$$W = \|\vec{F}\|\cos\theta\,\|\vec{d}\| = \boxed{\vec{F}\cdot\vec{d}.}$$

What about work done by force field $\vec{F}$ on particle moving along curve C?



Like before, use parametrization $\vec{r}(t)$.

From t_0 to t_1, work is roughly

$$\vec{F}(\vec{r}(t_0))\cdot(\vec{r}(t_1)-\vec{r}(t_0)) \approx \vec{F}(\vec{r}(t_0))\cdot\vec{r}\,'(t_0)(t_1-t_0)$$

Take sum, get $\text{Work} = \int_{t_0}^{t_{end}} \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt.$

Can rewrite this as $\text{Work} = \int_C \vec{F}\cdot \underbrace{d\vec{r}}_{\text{symbol for } \vec{r}\,'(t)dt}$

Or $\text{Work} = \int_{t_0}^{t_{end}} \vec{F}(\vec{r}(t))\cdot \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}\underbrace{\|\vec{r}\,'(t)\|dt}_{ds}$

unit tangent vector to curve

$$= \int_C \vec{F}\cdot\vec{T}\,ds$$

Ex Compute line integral $\int_C \vec{F} \cdot d\vec{r}$ where

$\vec{F}(x,y) = (2xy, x^2)$ $\vec{r}(t) = (t, t^3)$ cubic curve.
$0 \le t \le 1$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 F(t, t^3) \cdot (1, 3t^2)\, dt$$
$$= \int_0^1 2t^4 + 3t^4\, dt = t^5 \Big|_0^1 = 1.$$

Does not depend on choice of $\vec{r}(t)$ except for "orientation". Going backwards along C gives $(-1)\cdot$ answer.

Reparametrize $\vec{r}(t)$ backwards: just reverse time!

$\vec{r}(1-t)$ when $t=0$, get $\vec{r}(1)$
when $t=1$, get $\vec{r}(0)$

$\vec{q}(t) = \vec{r}(1-t) = (1-t, (1-t)^3)$

Now $\int_0^1 \vec{F}(1-t, (1-t)^3) \cdot (-1, -3(1-t)^2)\, dt$

$$= \int_0^1 (2(1-t)^4, (1-t)^2) \cdot (-1, -3(1-t)^2)\, dt$$
$$= \int_0^1 -5(1-t)^4\, dt \qquad \begin{aligned} u &= 1-t \\ du &= -dt \end{aligned}$$
$$= \int_1^0 5u^4\, du = u \Big|_1^0 = -1.$$

So $\int_C \vec{F} \cdot d\vec{r}$ depends (up to sign) on "orientation" of C. (Not true for $\int_C f\,ds$)

Another expression for $\int_C \vec{F} \cdot d\vec{r}$:

Write $\vec{F}(x,y) = (\underbrace{P(x,y)}_{\text{scalar}}, \underbrace{Q(x,y)}_{\text{functions}})$

Then $\displaystyle\int_C \vec{F}\cdot d\vec{r} = \int_{t_0}^{t_{end}} (P,Q)\cdot(x'(t), y'(t))\,dt$

$\displaystyle = \int_{t_0}^{t_{end}} \left[P(x,y)\,x'(t) + Q(x,y)\,y'(t)\right] dt$

$\displaystyle = \int_{t_0}^{t_{end}} P(x,y) \underbrace{x'(t)\,dt}_{dx} + \int_{t_0}^{t_{end}} Q(x,y) \underbrace{y'(t)\,dt}_{dy}$

$\displaystyle = \int_{x(t_0)}^{x(t_{end})} P\,dx + \int_{y(t_0)}^{y(t_{end})} Q\,dy$

(short hand) "$=$" $\displaystyle\int_C P\,dx + Q\,dy$.

Similarly, if $\vec{F}(x,y,z) = (P,Q,R)$ then

$$\int_C \vec{F}\cdot d\vec{r} = \int_C P\,dx + Q\,dy + R\,dz.$$

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§ 16.3 The Fundamental Theorem.

Review 1 var Calc, Fundamental Theorem says

For f continuous on $[a,b]$

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

Consequence $\displaystyle\int_a^x F'(t)\,dt = F(x) + C$

(have same derivative)

What is C? Plug in $x=a$, get

$0 = F(a) + C$, so $C = -F(a)$.

So $\int_a^x F'(t)\,dt = F(x) - F(a)$.

For line integrals?

1. $\int_C f\,ds$ Given parametrization $\vec{r}(s)$ of C by arc length, same FTC.

2. More interesting, $\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot\vec{T}\,ds$

Version 1: $\frac{d}{d\ell}\int_0^\ell \vec{F}(\vec{r}(s))\cdot\vec{T}(s)\,ds$
$$= \vec{F}(\vec{r}(\ell))\cdot\vec{T}(\ell)$$

or Version 2: $\int_C \nabla(f)\cdot d\vec{r} = f(\vec{r}(t_{end})) - f(\vec{r}(t_0))$.

Assume $\nabla(f)$ is continuous

Why?

$$\int_C \nabla(f)\cdot d\vec{r} = \int_{t_0}^{t_{end}} \left(\frac{\partial f}{\partial x}(\vec{r}(t)), \frac{\partial f}{\partial y}(\vec{r}(t))\right)\cdot \vec{r}\,'(t)\,dt$$
$$= \int_{t_0}^{t_{end}} \left[\frac{\partial f}{\partial x}(\vec{r}(t))\,x'(t) + \frac{\partial f}{\partial y}(\vec{r}(t))\,y'(t)\right] dt$$
$$= \int_{t_0}^{t_{end}} (f\circ\vec{r})'(t)\,dt \underset{\text{Old FTC}}{=} f(\vec{r}(t_{end})) - f(\vec{r}(t_0))$$

<u>Ex</u> $\vec{F}(x,y) = (2xy, x^2) = \nabla f$, where $f(x,y) = x^2 y$.

So C the cubic curve $\vec{r}(t) = (t, t^3)$,

have $\int_C \vec{F}\cdot d\vec{r} = f(1,1) - f(0,0) = 1 - 0 = 1$.

<u>Next Week</u> Conservative Vector Fields

- When is $\vec{F}$ conservative
- Why do we care?