Lecture 2

Aug. 24
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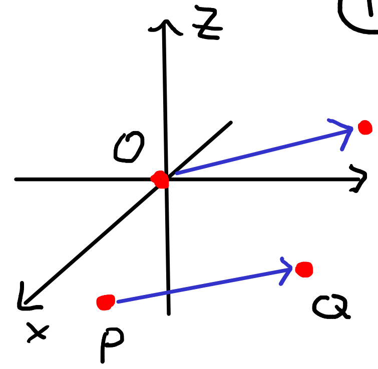
§ 12.2 Vectors

Points $P = (x_1, y_1, z_1)$
and $Q = (x_2, y_2, z_2)$
in $\mathbb{R}^3$ determine a <u>vector</u> $\overrightarrow{PQ}$



Define new point $R = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$

Then the vector $\overrightarrow{OR} = \overrightarrow{PQ}$.

<u>Upshot</u>: $\overrightarrow{PQ}$ completely described by the <u>point</u> R.

Often write vectors as $\vec{u}$, $\vec{v}$ (text uses a <u>bold</u> font)

<u>Vector operations</u>

1) Vector addition. $\vec{u}, \vec{v}$ vectors

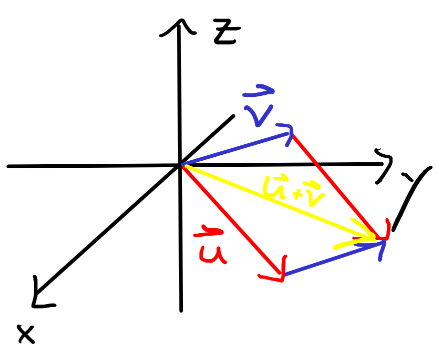
2 approaches:

i) <u>Algebraic</u>: write $\vec{u} = \overrightarrow{OP}$, $P = (x_1, y_1, z_1)$
$\vec{v} = \overrightarrow{OQ}$, $Q = (x_2, y_2, z_2)$

Often write $\vec{u} = (x_1, y_1, z_1)$ for shorthand

Then $\vec{u} + \vec{v} = \overrightarrow{OR}$, where
$R = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$

ii) <u>Geometric</u>



$\vec{u} + \vec{v}$ is the <u>diagonal</u> of the parallelogram.

Properties of Vector Addition

1. Associative: $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$
2. Commutative: $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
3. Zero: $\vec{u} + \vec{0} = \vec{u} = \vec{0} + \vec{u}$

   zero vector $\vec{0} = \overrightarrow{OO}$

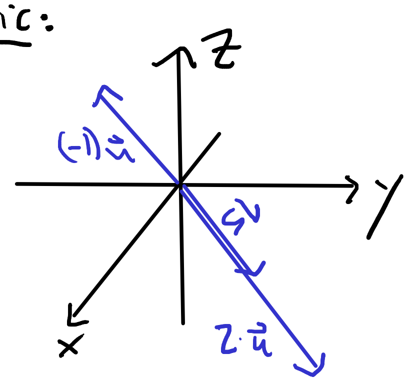
2) Scalar Multiplication: $\vec{u}$ vector, r scalar (number)

Can scale $\vec{u}$ by factor of r to get $r \cdot \vec{u}$.

i) Algebraic: write $\vec{u} = \overrightarrow{OP}$, $P = (x_1, y_1, z_1)$

Then $r \cdot \vec{u} = \overrightarrow{OR}$, where $R = (rx_1, ry_1, rz_1)$.

ii) Geometric:



Properties of Scalar Mutiplication

1. $(r \cdot s) \cdot \vec{u} = r \cdot (s \cdot \vec{u})$
2. Distributive Law: $r \cdot (\vec{u} + \vec{v}) = r \cdot \vec{u} + r \cdot \vec{v}$
3. $0 \cdot \vec{u} = \vec{0}$

Define the length (or norm or magnitude) of vector $\vec{u}$ to be distance from O to P, if $\vec{u} = \overrightarrow{OP}$. Write this as $|\vec{u}|$ or $\|\vec{u}\|$.

Properties
1. $\|r \cdot \vec{u}\| = |r| \, \|\vec{u}\|$
2. $\|\vec{u}\| = 0$ if and only if $\vec{u} = \vec{0}$.

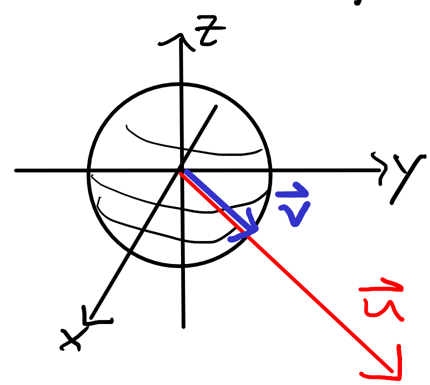
Say $\vec{u}$ is a <u>unit vector</u> if $\|\vec{u}\| = 1$.
(This means P lies on the unit sphere).

<u>Fact</u> Every nonzero vector $\vec{u}$ is a positive scalar multiple of a <u>unique</u> unit vector.

<u>Why?</u> Define $\vec{v} = \frac{1}{\|\vec{u}\|} \cdot \vec{u}$. (Note that $\|\vec{u}\| \neq 0$ since $\vec{u} \neq \vec{0}$)

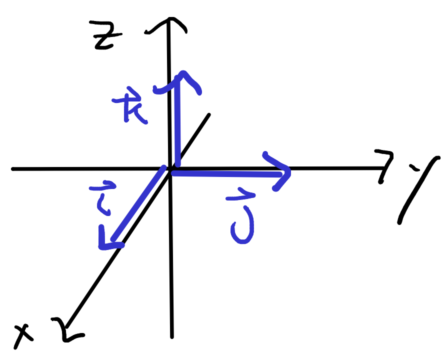
Then $\|\vec{v}\| = \left|\frac{1}{\|\vec{u}\|}\right| \cdot \|\vec{u}\| = 1$.

So $\vec{v}$ is a unit vector.



In addition to the zero vector $\vec{0}$, three special vectors:

$\vec{i} = (1,0,0)$
$\vec{j} = (0,1,0)$
$\vec{k} = (0,0,1)$



(Also called $\vec{e}_1, \vec{e}_2, \vec{e}_3$)

<u>Fact</u>: Can write any $\vec{u}$ in terms of $\vec{i}, \vec{j}, \vec{k}$.

If $\vec{u} = (u_1, u_2, u_3)$ (again, shorthand)

then $\vec{u} = u_1 \cdot \vec{i} + u_2 \vec{j} + u_3 \cdot \vec{k}$.

Collection $\{\vec{i}, \vec{j}, \vec{k}\}$ known as <u>standard basis</u>

§12.3 The Dot Product

We can add two vectors & multiply a number and a vector. Can we <u>multiply vectors</u>?

One way is by the <u>dot product</u> (a.k.a. <u>inner product</u>)

$$\vec{u}\cdot\vec{v} = \begin{cases} u_1v_1 + u_2v_2 & \text{for } \vec{u},\vec{v} \text{ in } \mathbb{R}^2 \\ u_1v_1 + u_2v_2 + u_3v_3 & \text{for } \vec{u},\vec{v} \text{ in } \mathbb{R}^3 \end{cases}$$

(Similarly for vectors in $\mathbb{R}^n$)

<u>Properties of dot product</u>

(1) Commutative: $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$

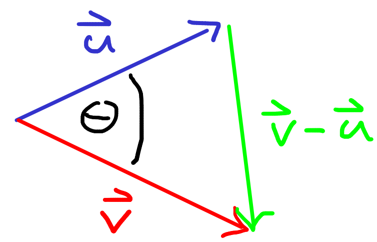
(2) Bilinear: a) $(\vec{u}+\vec{v})\cdot\vec{w} = \vec{u}\cdot\vec{w} + \vec{v}\cdot\vec{w}$

b) $(r\cdot\vec{u})\cdot\vec{w} = r(\vec{u}\cdot\vec{w})$

<u>Note</u>: $\vec{u}\cdot\vec{u} = u_1^2 + u_2^2 + u_3^2 = \|\vec{u}\|^2$, so

$$\|\vec{u}\| = \sqrt{\vec{u}\cdot\vec{u}}$$

<u>Connection to geometry</u>:



<u>Fact</u>: $\vec{u}\cdot\vec{v} = \|\vec{u}\|\cdot\|\vec{v}\|\cos\theta$

Comes from <u>Law of Cosines</u>

$$\underbrace{\|\vec{v}-\vec{u}\|^2}_{} = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\,\|\vec{v}\|\cos\theta$$

Now expand using bilinearity of the dot product.