Lecture 20

Oct. 10 2011

Last time: Fundamental Theorem for Line Integrals

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

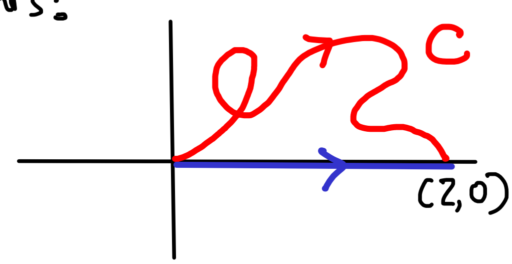
(Assume C smooth, ∇f continuous)

This week:
- Conservative Vector Fields (Mon & Wed)
- Double & Iterated Integrals

Exam 1: Tues Oct 18 @ 7 PM
Covers 14.6–14.8, Ch. 13, 16.1–16.3

Fundamental Theorem: $\int_C \nabla(f) \cdot d\vec{r} = f(\text{endpoint}) - f(\text{initial point})$.

So only depends on endpoints!



Ex $f = xy^2$

Find $\int_C \nabla f \cdot d\vec{r}$.

Take C_2 = line segment $(0,0) \longrightarrow (2,0)$.

Then $\int_C \nabla f \cdot d\vec{r} = \int_{C_2} \nabla f \cdot d\vec{r} = \int_0^1 \underbrace{\langle y(t)^2, 2x(t)y(t)\rangle}_{=\vec{0}} \cdot \vec{r}\,'(t)\,dt$

$= 0.$

More general statement:

If $F = \nabla(f)$, some f (F is "conservative")

then $\int_C F \cdot d\vec{r}$ is "path independent". (Only depends on endpoints).

Important case:

If F is conservative and C starts & ends at <u>same point</u> (say C is a "closed" curve or "loop") then

$$\int_C F \cdot d\vec{r} = 0.$$

<u>Ex</u> $F(x,y) = (-y, x)$ $\quad C$ = unit circle (counter-clockwise)

$$\vec{r}(t) = (\cos t, \sin t)$$

Then

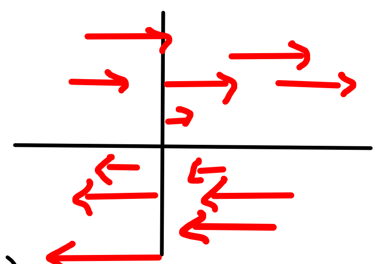
$$\int_C F \cdot d\vec{r} = \int_0^{2\pi} (-y(t), x(t)) \cdot (-\sin t, \cos t)\, dt$$

$$= \int_0^{2\pi} [-\sin(t)]^2 + [\cos t]^2\, dt = \int_0^{2\pi} dt = 2\pi.$$

Since $\int_C F \cdot d\vec{r} = 2\pi \neq 0$, F can't be conservative.

<u>Ex</u> $F(x,y) = (y, 0)$

Same C.



$$\int_C F \cdot d\vec{r} = \int_0^{2\pi} (\sin t, 0) \cdot (-\sin t, \cos t)\, dt$$

$$= -\int_0^{2\pi} \sin^2 t\, dt$$

Use Half angle identity $\dfrac{1 - \cos 2t}{2}$

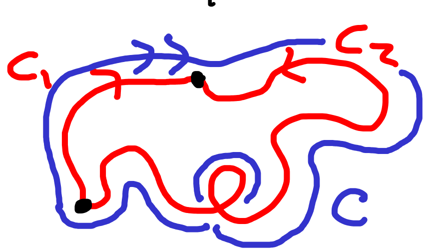
$$= \int_0^{2\pi} \frac{\cos 2t}{2} - \frac{1}{2}\, dt$$

$$= \frac{\sin 2t}{4} - \frac{t}{2} \Big|_0^{2\pi} = -\pi.$$

Not 0, so F not conservative.

Actually path independence $\longleftrightarrow$ vanishing on closed loops.

<u>Why?</u>



$$\int_{C_1} F\cdot d\vec{r} = \int_{C_2} F\cdot d\vec{r} \iff \int_{C_1} F\cdot d\vec{r} - \int_{C_2} F\cdot d\vec{r} = 0$$

$$\iff \int_{C_1} F\cdot d\vec{r} + \int_{-C_2} F\cdot d\vec{r} = 0$$

Reverse orientation

glue paths together

$$\iff \int_{C} F\cdot d\vec{r} = 0$$

Given conservative F, how to find f such that $\nabla f = F$?

<u>Ex</u> $F = \left(\frac{y^2}{1+x^2} - 1,\ 2y\arctan x + 3\right)$

Want $\frac{\partial f}{\partial x} = \frac{y^2}{1+x^2} - 1$, $\frac{\partial f}{\partial y} = 2y\arctan x + 3$.

$f = \int \frac{y^2}{1+x^2} - 1\, dx = y^2\arctan x - x + \underbrace{C}_{\text{constant (when } y \text{ is fixed)}}$

C may depend on y, so write $C = g(y)$.

$$\frac{\partial}{\partial y}\left[y^2\arctan x - x + g(y)\right] \overset{?}{=} 2y\arctan x + 3$$

$$\| $$

$$2y\arctan x + g'(y)$$

$\Rightarrow g'(y) = 3$, so $g(y) = 3y + D$ constant.

Now $\nabla(y^2\arctan x - x + 3y + D) = F$ for any D.

Free to pick $D = 0$. (Any choice is fine.)

How to tell if F is conservative?

2 Main Results:

First, some terminology.

D a region in $\mathbb{R}^2$.

Say D is <u>open</u> if whenever (x,y) is in D, can find small $\varepsilon > 0$ so that disc of radius ε with center (x,y) is contained in D.

<u>Open regions</u>

$\{(x,y) \mid x^2+y^2<r^2\}$

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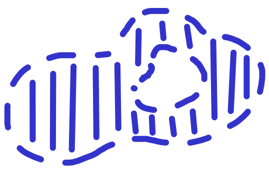
2 

<u>Not Open regions</u>

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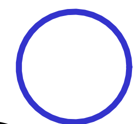
2 

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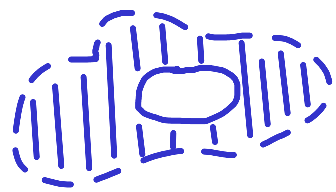
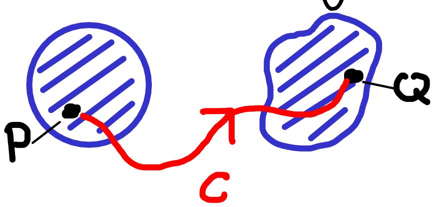
4 

Say D is <u>connected</u> if whenever P & Q are in D, there is a curve C in D joining P to Q.



<u>Not</u> connected.

<u>First Recognition Theorem</u>: F continuous vector field on open, connected region D in $\mathbb{R}^2$. Then

$\int_C F \cdot d\vec{r}$ is path-independent in D $\longleftrightarrow$ F is conservative on D.

<u>Why?</u> Assuming path-independence, define potential function $f(x,y)$ by $\int_{(a,b)}^{(x,y)} F \cdot d\vec{r}$ (pick any (a,b) in D).