Lecture 21

Oct. 12 2011

Last time:

First Recognition Theorem: F continuous vector field on open, connected region D in $\mathbb{R}^2$. Then

$\int_C F \cdot d\vec{r}$ is path-independent in D $\longleftrightarrow$ F is conservative on D.

Not practical! Today: More useful recognition result.

Quiz tomorrow

Exam 1: Tues Oct 18 @ 7 PM

Covers 14.6 - 14.8, Ch. 13, 16.1 - 16.3

Next idea Suppose $\vec{F} = \nabla f$.

$\vec{F} = P\vec{i} + Q\vec{j}$

By Clairaut, know (if 2nd partials of f are continuous)

$$\frac{\partial}{\partial y} P = \frac{\partial}{\partial y} \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} \frac{\partial f}{\partial y} = \frac{\partial}{\partial x} Q.$$

So conservative vector field $F = P\vec{i} + Q\vec{j}$ must satisfy $\frac{\partial}{\partial y} P = \frac{\partial}{\partial x} Q$.

Ex $F = (-y, x)$. $\frac{\partial}{\partial y}(-y) = -1 \neq \frac{\partial}{\partial x}(x) = 1$

F is not conservative.

Ex $F = (y, 0)$ $\frac{\partial}{\partial y}(y) = 1 \neq \frac{\partial}{\partial x}(0) = 0$

F is not conservative.

Ex $F = \left(\frac{y^2}{1+x^2} - 1,\ 2y\arctan x + 3\right)$

$$\frac{\partial}{\partial y}\left[\frac{y^2}{1+x^2} - 1\right] = \frac{2y}{1+x^2}$$
$$\|$$
$$\frac{\partial}{\partial x}\left[2y\arctan x + 3\right] = \frac{2y}{1+x^2}.$$

So F *may* be conservative. (Saw last time it *is*)

Question: When is this criterion enough?

Not always

Ex $\vec{F} = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$.

Then $\frac{\partial}{\partial y} P = \frac{(x^2+y^2)(-1) - (-y)(2y)}{[x^2+y^2]^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$

and $\frac{\partial}{\partial x} Q = \frac{x^2+y^2 - x(2x)}{[x^2+y^2]^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$.

But F is not conservative.

To see this, consider C unit circle.

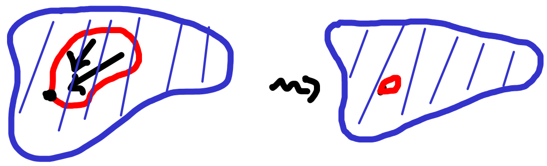
Then $\int_C F \cdot d\vec{r} = \int_0^{2\pi} \left(\frac{-\sin\theta}{1}, \frac{\cos\theta}{1}\right) \cdot (-\sin\theta, \cos\theta)\, d\theta$

$$= \int_0^{2\pi} [-\sin\theta]^2 + [\cos\theta]^2\, d\theta = 2\pi.$$

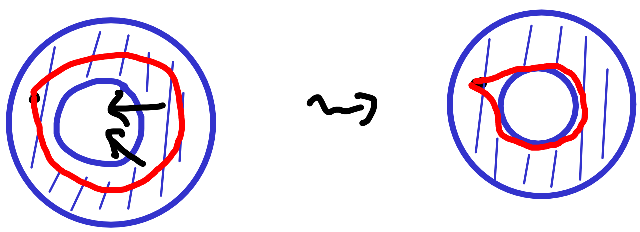
$\int_C F \cdot d\vec{r} \neq 0$, so F not conservative.

Need one more definition:

Connected region D (or Ω) is "simply connected" if every loop in D can be pulled to length 0 loop, keeping a pivot point fixed.



← Simply connected



← Not Simply Connected

Stuck!

Essentially just means no "holes" in domain.

Thm Second Recognition Theorem

Suppose $F = P\vec{i} + Q\vec{j}$, P & Q have continuous partials on open simply connected region D.

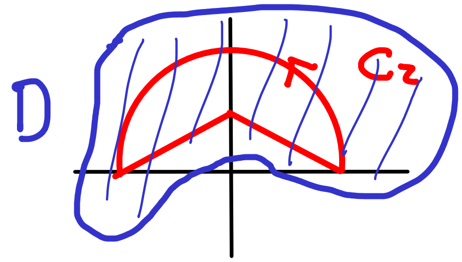
Then $\frac{\partial}{\partial y} P = \frac{\partial}{\partial x} Q \Longleftrightarrow F$ is conservative.

What goes wrong in previous example?

P & Q only defined on $\mathbb{R}^2 \setminus \{(0,0)\}$.

Not simply connected!

So, if we restrict the domain to $\{(x,y) \mid y > 0\}$, then F is conservative.



Ex C_2 is in a simply-conn. region in which P & Q behave nicely. So $\int_{C_2} F \cdot d\vec{r}$ should be 0.

Know from before $\int_{\text{semicircle}} F \cdot d\vec{r} = \pi$.

Segment from $(1,0)$ to $(0, \frac{1}{2})$ param by $\vec{r}(t) = (1,0) + t(-1, \frac{1}{2})$
$= (1-t, t/2)$.

Calculate $\int_{\text{segment}} F \cdot d\vec{r} = \int_0^1 \left(\frac{-t/2}{r^2}, \frac{1-t}{r^2} \right) \cdot (-1, 1/2)\, dt$

$= \cdots = \arctan(5/2\, t - 2) \Big|_0^1 = \arctan(\frac{1}{2}) - \arctan(-2)$

Also, $\int_{\text{other segment}} F \cdot d\vec{r} = \int_0^1 \left(\frac{\frac{t-1}{2}}{r^2}, \frac{-t}{r^2} \right) \cdot (-1, -\frac{1}{2})\, dt$

use $\vec{r}(t) = (-t, \frac{1-t}{2})$

$= \cdots = -\arctan(\frac{1}{2} - \frac{5}{2}t) \Big|_0^1 = -\arctan(-2) + \arctan(\frac{1}{2})$.

Add together, get $2\arctan(\frac{1}{2}) - 2\arctan(-2) = \pi$!

Why are these called "conservative" vector fields?

Assume $\vec{F} = \nabla f$ (f potential energy function)

(Actually, switch from math and physics, $f = -P$).

By Newton's 2^{nd} law, $\vec{F} = m\vec{a}(t)$
$= m\vec{r}\,''(t)$.

Recall kinetic energy $= \frac{1}{2}mv^2 = \frac{1}{2}m \|\vec{r}\,'(t)\|^2$.

Total energy = kinetic + potential.

$$E(t) = \frac{1}{2} m \|\vec{r}\,'(t)\|^2 + P(\vec{r}(t)).$$

Then $E'(t) = \frac{d}{dt}\left\{\frac{1}{2} m \| r'(t)\|^2\right\} + \frac{d}{dt}\left[P(\vec{r}(t))\right]$

$$= \frac{1}{2} m \frac{d}{dt}\left[x'(t)^2 + y'(t)^2\right] + \nabla(P) \cdot \vec{r}\,'(t)$$

$$= \frac{1}{\cancel{2}} m \left[\cancel{2} x'(t) x''(t) + \cancel{2} y'(t) y''(t)\right] + \nabla(P) \cdot \vec{r}\,'(t)$$

$$= m\, \vec{r}\,''(t) \cdot \vec{r}\,'(t) + \nabla(P) \cdot \vec{r}\,'(t)$$

$$= \vec{F} \cdot \vec{r}\,'(t) - \vec{F} \cdot \vec{r}\,'(t) = 0.$$

Total energy is <u>constant</u>