Lecture 22

Oct. 14
2011
1

Last time:

Exam 1: Tues Oct 18 @ 7 PM
Covers 14.6 - 14.8, Ch. 13, 16.1 - 16.3

Nathan (DD1 & DD6) 116 Roger Adams Lab
Yi (DD2 & DD3) 1404 Siebel
Michael & Caglar (DD4, DD5, & DD7) 1320 DCL

Monday OH 11:00 - 12:30
(not Friday)

First Recognition Theorem: F continuous vector field on open, connected region D in $\mathbb{R}^2$. Then

$\int_C F \cdot d\vec{r}$ is path-independent in D $\longleftrightarrow$ F is conservative on D.

Works exactly the same for $F: \mathbb{R}^3 \to \mathbb{R}^3$
or $F: \mathbb{R}^n \to \mathbb{R}^n$

Second Recognition Theorem $F = P\vec{i} + Q\vec{j}$, P & Q have continuous partials on open simply connected domain.

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \longleftrightarrow F \text{ conservative.}$$

Generalization to $F: \mathbb{R}^3 \to \mathbb{R}^3$? $F = P\vec{i} + Q\vec{j} + R\vec{k}$

Need $P_y = Q_x$ and $P_z = R_x$ and $Q_z = R_y$

(say "curl" $F = \vec{0}$. Will return to this in §16.5)

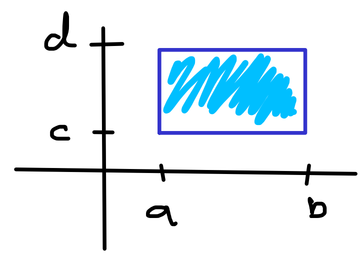
Next goal "Green's theorem". First need "double integrals".

---

§15.1 Double Int's.

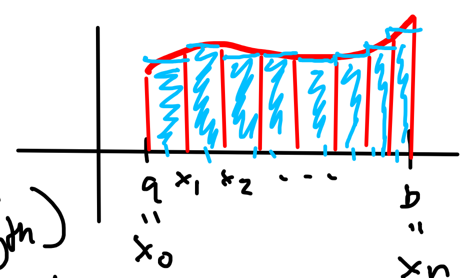
Goal: Given function $f: \mathbb{R}^2 \to \mathbb{R}$ use integration to measure volume of region under graph.

For simplicity, consider f on rectangle $[a,b] \times [c,d]$



First, review $\int_a^b f\,dx$.

Divide $[a,b]$ into small subintervals $[x_1, x_2]$ (equal length)



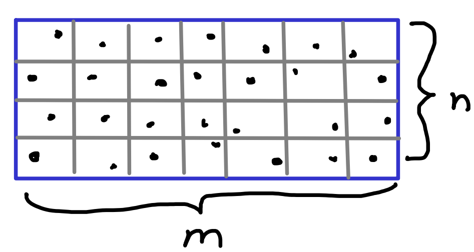
In $[x_{i-1}, x_i]$ take sample point x_i^*.

Approximate area by $f(x_1^*)\underbrace{(x_1 - x_0)}_{(b-a)/n} + f(x_2^*)\underbrace{(x_2 - x_1)}_{(b-a)/n} + \cdots + f(x_n^*)(x_n - x_{n-1})$.

$\int_a^b f(x)\,dx := \lim_{n \to \infty}$ above sum. ("Riemann sum").

Same technique for $f: \mathbb{R}^2 \to \mathbb{R}$.

Subdivide rectangle $R = [a,b] \times [c,d]$ by subdividing $[a,b]$ and $[c,d]$.



Take one sample point (x_{25}^*, y_{25}^*) in $R_{25} = [x_1, x_2] \times [y_4, y_5]$.

Consider double summation $\displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \underbrace{\Delta A}_{\frac{b-a}{m}\cdot\frac{d-c}{n} = \Delta x \Delta y}$

Define <u>double integral</u>

$$\iint_R f\, dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A$$

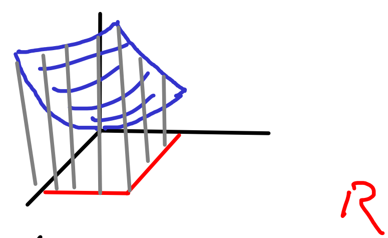
This measures the <u>volume</u> of the solid under graph of f, over R.

But how do we compute it?

§ 15.2 Reduce to single-variable integrals by considering "slices".

<u>Ex</u> $R = [0,2]\times[0,1]$, $f(x,y) = x^2+y^2$

Find $\iint_R f(x,y)\, dA$.



"Add up" area of "slices" where y is fixed.

$A(y) =$ area of y-slice $= \int_0^2 x^2+y^2\, dx = \frac{x^3}{3} + xy^2 \Big|_0^2 = \frac{8}{3} + 2y^2$.

Now add up all areas $\rightsquigarrow$ $\text{Vol} = \int_0^1 A(y)\, dy =$

$$= \int_0^1 \frac{8}{3} + 2y^2\, dy = \frac{8y}{3} + 2\frac{y^3}{3}\Big|_0^1 = 8/3 + 2/3 = 10/3.$$

<u>OR</u> we could first slice in x-direction, then take $\int_0^2 A(x)\,dx$.

$$A(x) = \int_0^1 x^2+y^2\, dy = x^2y + \frac{y^3}{3}\Big|_0^1 = x^2 + \frac{1}{3}.$$

$$\text{Vol} = \int_0^2 A(x)\,dx = \int_0^2 x^2 + \tfrac{1}{3}\, dx = \frac{x^3}{3} + \frac{x}{3}\Big|_0^2 = 8/3 + 2/3 = \frac{10}{3}.$$

Theorem (Fubini) Assume f continuous on $R=[a,b]\times[c,d]$

Then $$\iint_R f(x,y)\,dA = \int_a^b\int_c^d f(x,y)\,dy\,dx$$
$$= \int_c^d\int_a^b f(x,y)\,dx\,dy.$$

Known as "iterated integrals"

Note Sometimes, one formulation more convenient than other.

Ex $f(x,y)=\dfrac{x}{1+xy}$ $\quad R=[0,1]\times[0,1]$

Try $\int_0^1\int_0^1 \frac{x}{1+xy}\,dx\,dy$ means integrate first w.r.t. x.

Tricky!

Try instead $\int_0^1\int_0^1 \frac{x}{1+xy}\,dy\,dx$ $\quad$ let $u=1+xy$ (x constant)
$du = x\,dy$

$$=\int_0^1\int_1^{1+x}\frac{du}{u}\,dx = \int_0^1 \ln(1+x) - \cancel{\ln(1)}\,dx$$

Integrate by parts $\quad u=\ln(1+x) \quad v=1+x$
$du=\dfrac{dx}{1+x} \quad dv=dx$

$$= uv - \int v\,du$$

$$= \ln(1+x)\,(1+x)\Big|_0^1 - \int_0^1 \frac{1+x}{1+x}\,dx \quad = (\ln 2)2-(\ln 1)1 - 1-0$$
$$= 2\ln 2 - 1 = \ln 4 - 1$$

Next time $\iint_D f(x,y)\,dA$

general region.

Uses: 1) $\iint_D 1\,dA = \text{area}(D)$

2) average value of f on $D = \dfrac{1}{\text{area}(D)}\iint_D f(x,y)\,dA.$