Lecture 23

Oct. 17 2011

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Exam 1: Tues Oct 18 @ 7 PM

Covers 14.6-14.8, Ch. 13, 16.1-16.3

Nathan (DD1 & DD6) 116 Roger Adams Lab

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No Class Friday

Last time: Double (iterated integrals) over rectangle

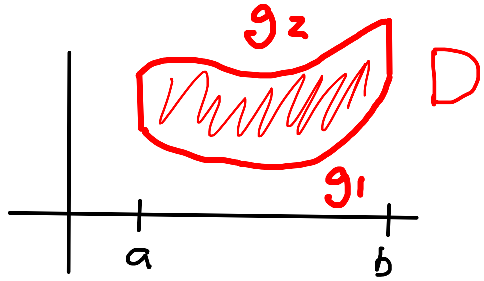
$R = [a,b] \times [c,d]$ then for f continuous on R

$$\iint_R f \, dA = \int_a^b \underbrace{\int_c^d f(x,y)\,dy}_{A(x)}\,dx = \int_c^d \underbrace{\int_a^b f(x,y)\,dx}_{A(y)}\,dy$$

What about more general domain D?

Example: Suppose D of the form

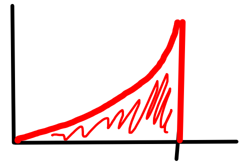
$$D = \left\{ (x,y) \,\middle|\, \begin{array}{l} a \le x \le b \\ g_1(x) \le y \le g_2(x) \end{array} \right\}$$



Then should still consider $\int_a^b A(x)\,dx$.

What is $A(x)$? $\quad A(x) = \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy$

<u>Ex</u> $D=\left\{(x,y) \;\middle|\; \begin{array}{l} 0\le x\le 1 \\ 0\le y\le x^2 \end{array}\right\}$



$f(x,y)=xy-y \qquad g_1(x)=0,\ g_2(x)=x^2$

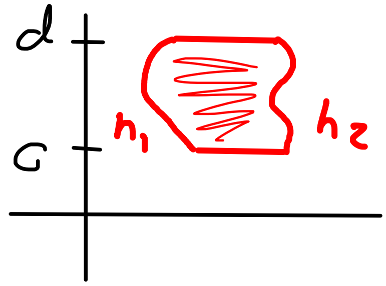
$$\iint_D xy-y\,dA=\int_0^1 A(x)\,dx=\int_0^1\left[\int_0^{x^2}(x-1)y\,dy\right]dx$$

$$=\int_0^1 (x-1)\frac{y^2}{2}\Big|_0^{x^2}dx=\frac{1}{2}\int_0^1 (x-1)x^4\,dx$$

$$=\frac{1}{2}\left(\frac{x^6}{6}-\frac{x^5}{5}\right)\Big|_0^1=\frac{1}{2}\left(\frac{1}{6}-\frac{1}{5}\right)=\frac{1}{2}\,\frac{-1}{30}=\frac{-1}{60}.$$

<u>Alternative</u> D given in form

$$D=\left\{(x,y) \;\middle|\; \begin{array}{l} c\le y\le d \\ h_1(y)\le x\le h_2(y) \end{array}\right\}$$



Then we can use $\iint_D f\,dA=\int_c^d A(y)\,dy$

Here $A(y)=\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx$.

So $\iint_D f\,dA=\int_c^d\left[\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\right]dy$

<u>Ex</u> Same example. $h_1(y)=\sqrt{y}$, $h_2(y)=1$.

$$\iint_D f\,dA=\int_0^1\left[\int_{\sqrt{y}}^{1} xy-y\,dx\right]dy$$

$$=\int_0^1 \frac{x^2y}{2}-xy\,\Big|_{\sqrt{y}}^{1}dy$$

$$=\int_0^1\left\{\frac{y}{2}-y\right\}-\left[\frac{y^2}{2}-y^{3/2}\right]dy$$

$$= \int_0^1 -\frac{y}{2} - \frac{y^2}{2} + y^{3/2}\, dy$$

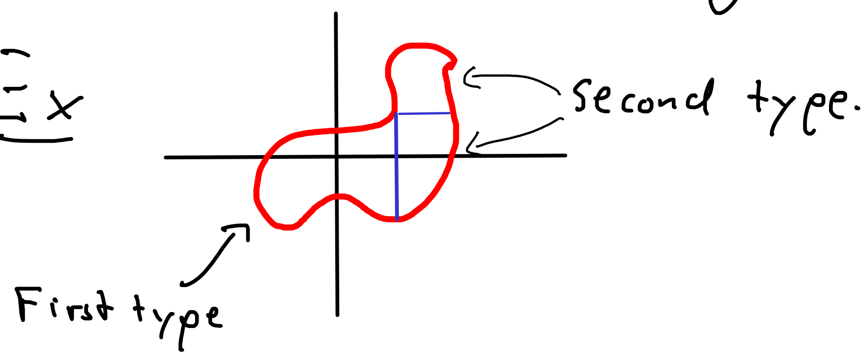
$$= -\frac{y^2}{4} - \frac{y^3}{6} + \frac{2y^{5/2}}{5} \Big|_0^1 = -\frac{1}{4} - \frac{1}{6} + \frac{2}{5}$$

$$= \frac{-15 - 10 + 24}{60} = \frac{-1}{60}.$$

Computed same volume in 2 different ways.

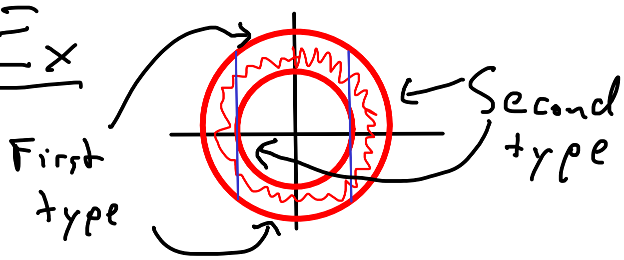
How do we handle more general regions?

Ex



Sometimes, another approach is more convenient.

Ex



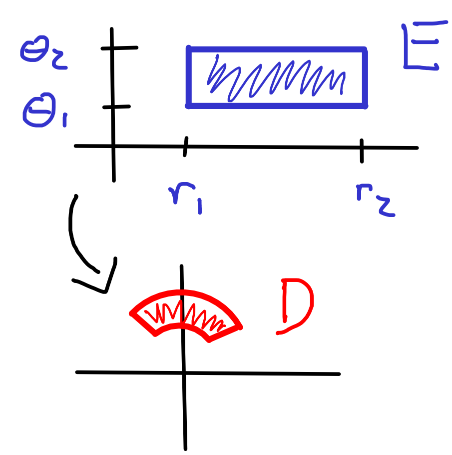
Try change of variables.

Use polar coordinates

Recall any (x,y) in $\mathbb{R}^2$ can be describe in terms of polar coordinates (r, θ)

$x = r\cos\theta$, $y = r\sin\theta$, $r = \sqrt{x^2+y^2}$

$\theta = \arctan\left(\frac{y}{x}\right)$ if $x \neq 0$.



How does a polar rectangle transform into cartesian coordinates?

Can we relate $\iint_D f\,dA$ to an integral in θ and r?

<u>Guess:</u> $\iint_D f\,dA = \iint_E f\,dr\,d\theta$?

Try with $f=1$, $E = \left\{ (r,\theta) \,\middle|\, \begin{array}{l} 1 \le r \le 2 \\ 0 \le \theta \le 2\pi \end{array} \right\}$

Then $D = \{(x,y) \mid 1 \le x^2+y^2 \le 4\}$

Know $\iint_D 1\,dA = \text{area}(D) = 4\pi - \pi = 3\pi$.

But $\int_0^{2\pi}\int_1^2 dr\,d\theta = \int_0^{2\pi} r\Big|_1^2 d\theta$

$= \int_0^{2\pi} d\theta = 2\pi$. ← Not the same!

<u>Trouble:</u> Change of coordinates distorts area.
(<u>Not</u> uniformly)

Polar rectangle $\left\{ \begin{array}{l} 1 \le r \le 2 \\ 0 \le \theta \le \frac{\pi}{2} \end{array} \right\}$ Area $\frac{\pi}{2}$ $\rightsquigarrow$ Portion of annulus

Area $= \frac{\text{angle}}{2\pi} \cdot$ area annulus

$= \frac{1}{4} \cdot 3\pi = \frac{3\pi}{4}$

$\boxed{\text{multiplied by } \frac{3}{2}}$

Polar rectangle $\left\{ \begin{array}{l} 2 \le r \le 3 \\ 0 \le \theta \le \frac{\pi}{2} \end{array} \right\}$ Area $\frac{\pi}{2}$ $\rightsquigarrow$ Portion of annulus

Area $= \frac{\text{angle}}{2\pi} \cdot$ area annulus

$= \frac{1}{4} \cdot 5\pi = \frac{5\pi}{4}$

$\boxed{\text{multiplied by } 5/2}$

In both cases, scaling factor is <u>average</u> <u>radius</u>.

New guess $\iint_D f\,dA = \iint_E f\;r\,dr\,d\theta$

$$\int_0^{2\pi}\int_1^2 r\,dr\,d\theta = \int_0^{2\pi} \frac{r^2}{2}\Big|_1^2\,d\theta$$

$$= \int_0^{2\pi} 2 - \tfrac{1}{2}\,d\theta$$

$$= 3/2 \cdot 2\pi = 3\pi.$$ Correct!