# Lecture 24

Oct. 19
2011
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No Class Friday

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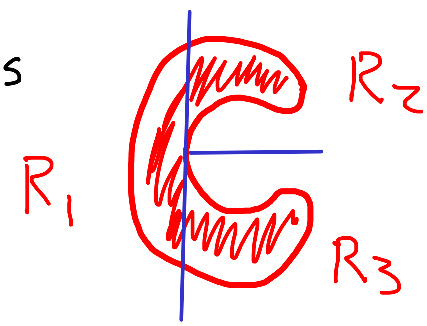
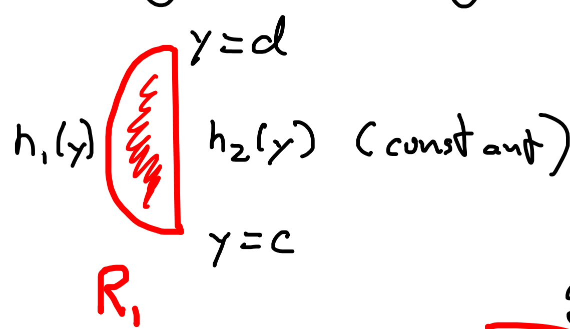
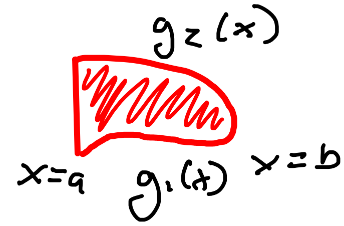
Last time: Double (iterated integrals) over general regions




$$\iint_R f\,dA = \iint_{R_1} f\,dA + \iint_{R_2} f\,dA + \iint_{R_3} f\,dA$$

$$\iint_{R_1} f\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

$$\iint_{R_2} f\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx.$$

Ended with polar coordinates:

Change of variables formula:

$$\iint_D f(x,y)\,dA = \iint_E f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

Example Compute volume of sphere of radius R.

Let $f = \sqrt{R^2 - x^2 - y^2}$, $D = \{(x,y) \mid x^2+y^2 \le R^2\}$.

Then Volume $= 2\iint_D f\,dA$

$$= 2\int_0^{2\pi}\int_0^R f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

$$= 2\int_0^{2\pi}\int_0^R \sqrt{R^2-(r\cos\theta)^2-(r\sin\theta)^2}\; r\,dr\,d\theta$$

$$= 2\int_0^{2\pi}\int_0^R \sqrt{R^2-r^2}\; r\,dr\,d\theta$$

$$= 4\pi \int_0^R \sqrt{R^2-r^2}\; r\,dr \qquad u = R^2-r^2, \quad du = -2r\,dr$$

$$= -2\pi \int_{R^2}^0 \sqrt{u}\,du$$

$$= -2\pi \left.\frac{2u^{3/2}}{3}\right|_{R^2}^{0} = \frac{4}{3}\pi R^3.$$

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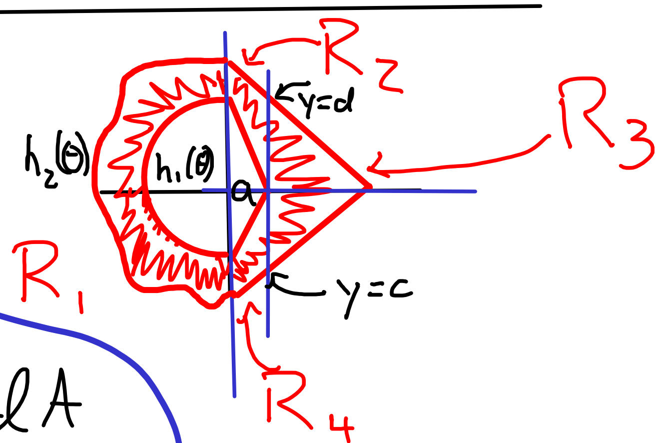
Mix & match: Region




$$\iint_R f\,dA = \iint_{R_1} f\,dA + \iint_{R_2} f\,dA + \iint_{R_3} f\,dA + \iint_{R_4} f\,dA$$

$$\int_{\pi/2}^{3\pi/2}\int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta \qquad \int_0^a \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

$$\int_c^d \int_{j_1(y)}^{j_2(y)} f(x,y)\,dx\,dy$$

$a = j_1(y)$

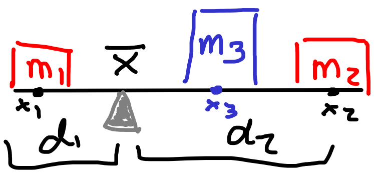
# §15.5 Some applications.

Already saw $\text{area}(R) = \iint_R 1 \, dA$.

So average value of $f$ on $R$ $= \dfrac{1}{\text{area}(R)} \iint_R f \, dA$.

Generalizes behavior on lines, curves.

**Center of mass**: First review 1-dim case.




"center of mass" $\bar{x}$ = place to put pivot (fulcrum) so that it's balanced.

Need $d_1 m_1 = d_2 m_2$. "moments"

$(\bar{x} - x_1) m_1 = (x_2 - \bar{x}) m_2$

$\bar{x}(m_1 + m_2) = x_1 m_1 + x_2 m_2 \implies \bar{x} = \dfrac{x_1 m_1 + x_2 m_2}{m_1 + m_2}$

Add 3rd mass $m_3$ at $x_3$

Then need $d_1 m_1 = d_2 m_2 + d_3 m_3$,

get $\bar{x}(m_1 + m_2 + m_3) = x_1 m_1 + x_2 m_2 + x_3 m_3$.

So $\bar{x} = \dfrac{x_1 m_1 + x_2 m_2 + x_3 m_3}{m_1 + m_2 + m_3}$. ← $\dfrac{\text{"total moment"}}{\text{"total mass"}}$

Above assumes massless beam.

Let $\rho(x)$ = density function for the beam.

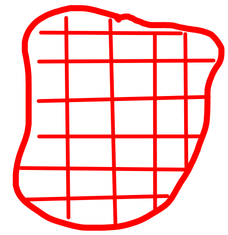
Center of mass $\bar{x} = \dfrac{1}{\text{total mass}} \overbrace{\int_a^b x \cdot \rho(x) \, dx}^{\text{moment}}$.

Generalize to 2 dimensions.

Flat region $R$ w/ density function $\rho(x,y)$.

Consider balance in $x$-direction & in $y$-direction.

<u>Moment in $x$-dir</u> sum up $x\cdot\begin{pmatrix}\text{mass of}\\ \text{rectangle}\end{pmatrix}$



aka "moment about $y$-axis" $\hookrightarrow M_x = \iint_R x\,\rho(x,y)\,dA$

Then $x$-coord of center of mass is $\bar{x} = \frac{1}{\text{total mass}} \iint_R x\,\rho(x,y)\,dA$

Similarly, get $M_y = \iint_R y\,\rho(x,y)\,dA$

and $\bar{y} = \frac{1}{\text{total mass}} \iint_R y\,\rho(x,y)\,dA$.

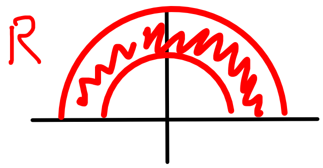
Center of mass (aka "centroid") $= (\bar{x}, \bar{y})$.

In case of constant density $\rho \equiv K$, then

$$\bar{x} = \frac{1}{\iint_R K\,dA} \iint_R x\,K\,dA = \frac{\cancel{K}}{\cancel{K}} \frac{1}{\text{area}(A)} \iint_R x\,dA$$

$=$ average $x$-coord in region $R$.

<u>Ex</u>




Inner radius $= 1$
Outer radius $= 2$ , constant density.
Find balance point. (may as well assume density $= 1$)

Total area $= \frac{1}{2}(4\pi - \pi) = \frac{3\pi}{2}$.

$M_x = 0$ since $R$ symmetric about $y$-axis.

$$M_y = \iint_R y\, dA = \int_0^{\pi}\int_1^{2} r\sin\theta \; r\,dr\,d\theta$$

does not depend on r

$$= \int_0^{\pi} \sin\theta \left[\int_1^{2} r^2\,dr\right] d\theta$$

does not depend on r

$$= \int_1^{2} r^2\,dr \cdot \int_0^{\pi} \sin\theta\, d\theta$$

$$= \left(\tfrac{8}{3} - \tfrac{1}{3}\right)\left(-\cos\pi + \cos 0\right) = 14/3$$

So $\bar{y} = \dfrac{1}{3\pi/2} \cdot \dfrac{14}{3} = \dfrac{28}{9\pi} \approx 0.99.$

So balance point lies <u>outside</u> the region!

This is impossible to balance.

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<u>Next week:</u> Triple integrals $\iiint_E f(x,y,z)\,dV$

- Volume$(E) = \iiint_E dV$
- change of coords to cylindrical or spherical coords