Lecture 26

Last time: Triple integrals $\iiint_R f\,dV$.

Main idea: take slice in one variable, integrate volume of f restricted to slices: $\int_a^b \left[\iint_D f\,dA\right] dx$

For double integrals, polar coordinates sometimes useful.

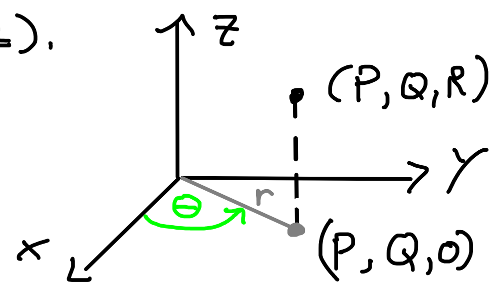
For triple integrals, 2 new coordinate systems.

Cylindrical coords Just replace x & y by polar.

$(x, y, z) = (r\cos\theta, r\sin\theta, z)$.

$r = \sqrt{x^2+y^2}$ $\tan\theta = y/x$

Useful for solids having symmetry around one axis (z-axis).



e.g. 1) Cylinder $x^2+y^2=d^2$ given by $r=d$ in cylindrical coords

2) Cone $x^2+y^2=z^2$ given by $r=|z|$.

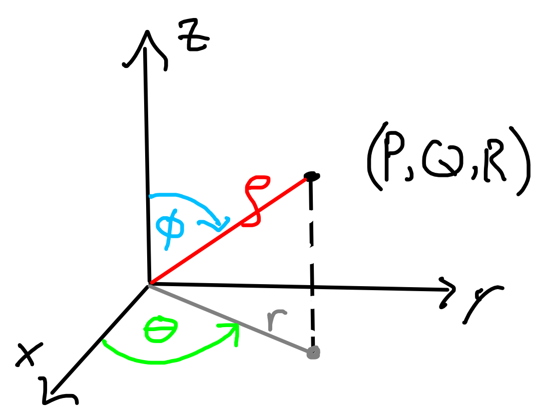
3) Elliptic paraboloid $x^2+y^2=z$ given by $r^2=z$.

Spherical

3 coordinates are $\rho \geq 0$ (rho)

theta $0 \leq \theta \leq 2\pi$

phi $0 \leq \phi \leq \pi$



$\rho = \sqrt{x^2+y^2+z^2}$

θ same as before $\tan\theta = y/x$

$z = \rho\cos\phi$, $r = \rho\sin\phi$ $\rightsquigarrow$ $x = r\cos\theta = \rho\sin\phi\cos\theta$

$\rightsquigarrow$ $y = r\sin\theta = \rho\sin\phi\sin\theta$

$$\tan\phi = \frac{r}{z} = \frac{\sqrt{x^2+y^2}}{z}$$

Ex Solid sphere of radius d:

Cartesian	Cylindrical	Spherical
$x^2+y^2+z^2 \le d^2$	$r^2+z^2 \le d^2$	$\rho^2 \le d^2$
$-d \le z \le d$	$-d \le z \le d$	or $\rho \le d$.
$-\sqrt{d^2-z^2} \le y \le \sqrt{d^2-z^2}$	$0 \le r \le \sqrt{d^2-z^2}$	$0 \le \phi \le \pi$
$-\sqrt{d^2-z^2-y^2} \le x \le \sqrt{d^2-z^2-y^2}$	$0 \le \theta \le 2\pi$	$0 \le \theta \le 2\pi$

How to integrate in these new coordinates?

Already know answer for cylindrical (same as for polar)

$$dV = r\,dr\,d\theta\,dz$$

Vol sphere radius d $= \displaystyle\int_{-d}^{d}\int_0^{2\pi}\int_0^{\sqrt{d^2-z^2}} r\,dr\,d\theta\,dz$

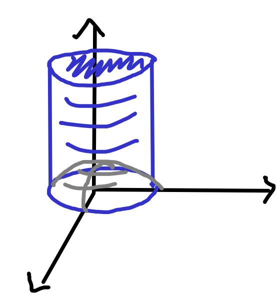
$$= 2\pi\int_{-d}^{d} \frac{d^2-z^2}{2}\,dz = 2\pi\left[\frac{d^2}{2}\cdot 2d - \left(\frac{d^3+d^3}{6}\right)\right]$$

$$= 2\pi\left(d^3 - \tfrac{1}{3}d^3\right) = \tfrac{4}{3}\pi d^3.$$

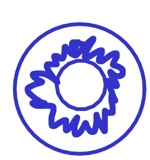
Another example (for cylindrical coords)

E = solid bounded by $z=4$,

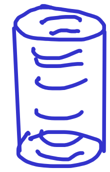
cylinder $x^2+y^2=1$ & paraboloid $z = 1-x^2-y^2$



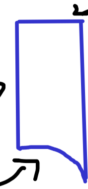
Can slice in 3 directions:

in z, look like  if $z \ge 1$ and like  if $z < 1$.

in r, look like



in θ, look like



Can set up the integral in 6 ways:

slicing in z: differing behavior for $z \geq 1$ or $z \leq 1$, so use 2 integrals

(1) $\int_1^4 \int_0^{2\pi} \int_0^1 r\,dr\,d\theta\,dz + \int_0^1 \int_0^{2\pi} \int_{\sqrt{1-z}}^1 r\,dr\,d\theta\,dz$

(2) $\int_1^4 \int_0^1 \int_0^{2\pi} r\,d\theta\,dr\,dz + \int_0^1 \int_{\sqrt{1-z}}^1 \int_0^{2\pi} r\,d\theta\,dr\,dz$

slicing in r:

(3) $\int_0^1 \int_0^{2\pi} \int_{1-r^2}^4 r\,dz\,d\theta\,dr$

(4) $\int_0^1 \int_{1-r^2}^4 \int_0^{2\pi} r\,d\theta\,dz\,dr$

slicing in θ:

(5) $\int_0^{2\pi} \int_0^1 \int_{1-r^2}^4 r\,dz\,dr\,d\theta$

(6) $\int_0^{2\pi} \left[\int_0^1 \int_{\sqrt{1-z}}^1 r\,dr\,dz + \int_1^4 \int_0^1 r\,dr\,dz \right] d\theta$

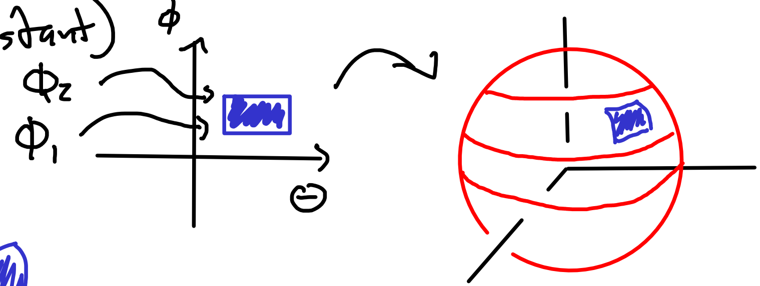
Any way you slice it, get $\frac{7\pi}{2}$.

How about spherical coords?

How much does this distort volume?

Start w/ rectangle in (θ, ϕ)-plane.

(ρ thought of as constant)



Vertical sides of  have length $\rho \Delta \phi$. Horizontal sides? Top one lies on circle $z = \rho \cos \phi_1$ w/ radius $\rho \sin \phi_1$

So top side has length $(\rho \sin \phi_1) \Delta \theta$

Bottom one has length $(\rho \sin \phi_2) \Delta \theta$.

So area approximated by $\rho \Delta \phi \cdot \rho \sin \phi \Delta \theta = \rho^2 \sin \phi \, \Delta \theta d\phi$,

Box in spherical coords w/ sides of length $\Delta \rho \cdot \Delta \theta \cdot \Delta \phi$

gives "wedge" of volume $\boxed{\rho^2 \sin \phi \, \Delta \rho \Delta \theta \Delta \phi := dV}$