Lecture 27

Oct. 27 2011

Quiz: Tuesday, on double & triple integrals

Last time: Integrating in cylindrical coords

$$\iiint f \, dx\,dy\,dz = \iiint f \; r\,dr\,d\theta\,dz$$

and in spherical coords

$$\iiint f \, dx\,dy\,dz = \iiint f \; \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi.$$

Ex Again the sphere of radius d

$$\text{Vol} = \int_0^{2\pi}\int_0^{\pi}\int_0^{d} \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= 2\pi \left[\int_0^{\pi} \sin\phi \, d\phi\right]\left[\int_0^{d} \rho^2 \, d\rho\right]$$

$$= 2\pi \cdot 2 \cdot \frac{d^3}{3} = \frac{4}{3}\pi d^3$$

Today & Monday: Change of coordinates

For single integrals usually rewrite $f(x) = h(u(x)) \cdot u'(x)$

Then $$\int_a^b f(x)\,dx = \int_a^b h(u(x))\,u'(x)\,dx$$

$$= \int_{u(a)}^{u(b)} h(u)\,du.$$

Or, parametrizing by $x = x(t)$, then

$$\int_a^b f(x)\,dx = \int_{t_0}^{t_1} f(x(t))\,x'(t)\,dt$$

For double integrals, given new coords $u(x,y)$ & $v(x,y)$ or parametrizations $x=x(s,t)$, $y=y(s,t)$

how to convert $\iint f\,dx\,dy$ into $\iint \boxed{?}\,du\,dv$?

or $\iint \boxed{?}\,ds\,dt$

Again, question is how much the transformation distorts area.

Start with easy "Linear Transformation"

$$x = 3s+t \quad , \quad y = 2s+3t.$$

Write $T(s,t) = (3s+t, 2s+3t)$.

Linear Transformations satisfy

(a) $T(0,0) = 0,0$

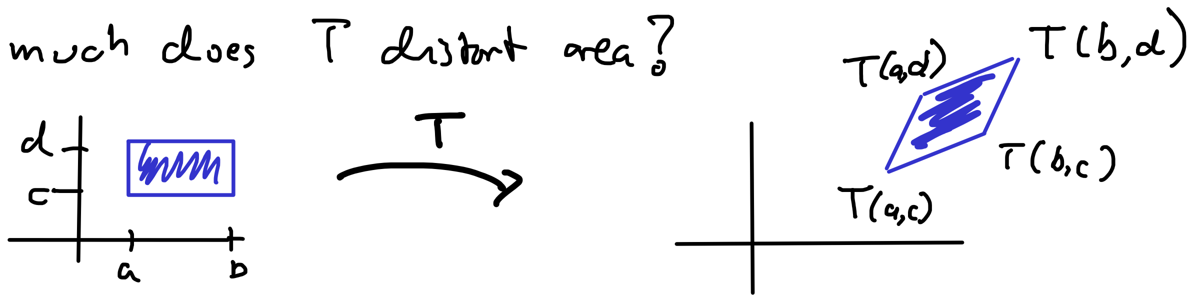
(b) $T(a,b) + T(c,d) = T(a+c, b+d)$

(c) If know $T(1,0)$ & $T(0,1)$, know

$T(s,t)$ for all s & t.

$$T(s,t) = s\,T(1,0) + t\,T(0,1)$$

How much does T distort area?



Orignal area $(b-a)(d-c)$.

New area: use cross product

$$= \|(T(b,c) - T(a,c)) \times (T(a,d) - T(a,c))\|$$

$$= \| T(b-a, 0) \times T(0, d-c) \|$$

$$= \underbrace{(b-a)(d-c)}_{\text{old area}} \underbrace{\| T(1,0) \times T(0,1) \|}_{\text{scaling factor, independent of } a,b,c,d}$$

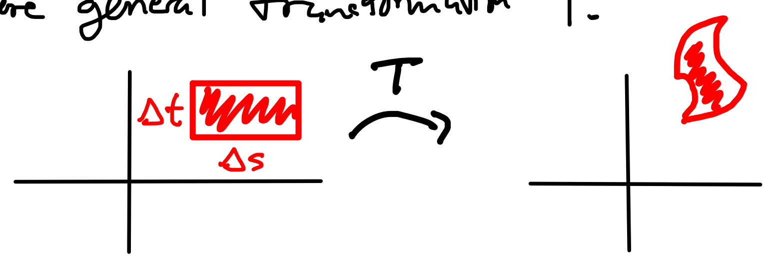
$$= (b-a)(d-c) \left\| \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 2 & 0 \\ 1 & 3 & 0 \end{vmatrix} \right\| = (b-a)(d-c) \begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix}$$

The "Jacobian" of T.

So $$\iint_D f(x,y)\, dx\, dy = \iint_E f(x(s,t), y(s,t)) \cdot 7 \, ds\, dT$$

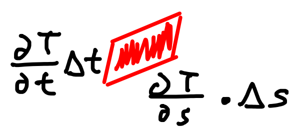
Use same idea for more general transformation T.

For small rectangle



Estimate by Linear approx, $T \approx T(s_0, t_0) + \frac{\partial T}{\partial s} \Delta s + \frac{\partial T}{\partial t} \Delta t$

Estimate area by parallelogram



with area $\Delta s \Delta t \cdot \left\| \frac{\partial T}{\partial s} \times \frac{\partial T}{\partial t} \right\|$, $T(s,t) = (x(s,t), y(s,t), 0)$

$$= \Delta s \Delta t \left\| \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & 0 \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & 0 \end{vmatrix} \right\| = \Delta s \Delta t \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{vmatrix}$$

The "Jacobian" of T

So $dx\,dy = J(T)\, ds\, dt$

and

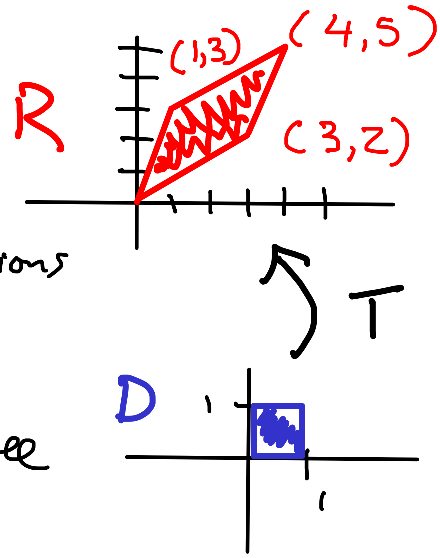
Also written $J(T) = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{vmatrix}$

$$\iint_D f(x,y)\,dx\,dy = \iint_E f(x(s,t),\, y(s,t))\, J(T)\, ds\,dt$$

__Ex__ T = polar transformation $T(r,\theta) = (r\cos\theta, r\sin\theta)$

Then $J(T) = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta, & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$

__Ex__ Find $\iint_R x+y\, dA$



Could slice into type I & II regions

__or__ use change of coords.

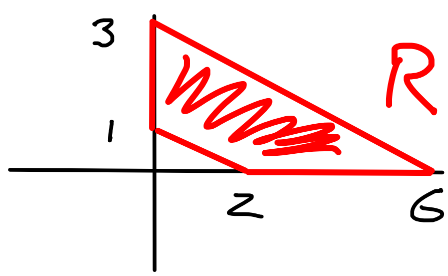
Using $T = (3s+t,\ 2s+3t)$ see

So $\iint_R x+y\, dA = \iint_D (3s+t)+(2s+3t)\ 7\, ds\,dt$

$= 7\int_0^1\int_0^1 5s+4t\, ds\,dt$

$= 7\int_0^1 5/2 + 4t\, dt = 7(5/2 + 4/2) = \frac{63}{2}.$

__Ex__ Find $\iint_R y\, dA$



How do change coords?

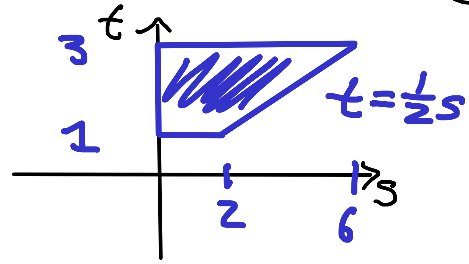
The lines are $y = 1 - \frac{1}{2}x$ & $y = 3 - \frac{1}{2}x$, $y + \frac{1}{2}x$ constant on both.

Would like $t = y + \frac{1}{2}x$ (keep $x = s$).

$t = y + \frac{1}{2}s$, so $y = t - \frac{1}{2}s$.

Have $T(s,t) = (s, t - \frac{1}{2}s)$.

$$J(T) = \begin{vmatrix} 1 & 0 \\ -1/2 & 1 \end{vmatrix} = \underline{1}.$$



So $\iint_R y \, dA = \int_1^3 \int_0^{2t} t - \frac{1}{2}s \, ds\, dt = \cdots = \frac{26}{3}$.

Next time Change of coords in $\underline{3}$ variables.