Lecture 28

Oct. 31 2011

(1)

<u>Quiz</u>: Tuesday, on double & triple integrals

This Friday's Lecture: in 100 Noyes

<u>Last time</u>: Change of coordinates for double integrals.

Given transformation $T : (s,t)$-plane $\to$ (x,y)-plane

define the <u>Jacobian</u> of T by

$$J(T) = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{vmatrix}$$

Then

$$\iint_D f(x,y)\,dx\,dy = \iint_E f(x(s,t), y(s,t))\,|J(T)|\,ds\,dt$$

Where T transforms E to D. $(D = T(E))$

<u>Note</u>: need transformation T to be <u>one-to-one</u>:

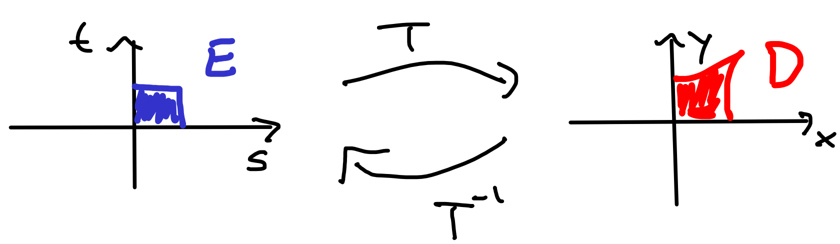
if $T(s_1,t_1) = T(s_2,t_2)$ then $s_1 = s_2$ and $t_1 = t_2$.

<u>Example</u> Polar transformation $T(r,\theta) = (r\cos\theta, r\sin\theta)$

is <u>not</u> one-to-one on $E = \{(r,\theta) \mid -2\pi \le \theta \le 2\pi,\ 0 \le r \le 1\}$

$$\int_{-2\pi}^{2\pi}\int_0^1 r\,dr\,d\theta = \underline{\underline{2}}\,\pi r^2, \text{ not } \pi r^2.$$

If T is one-to-one, there is <u>inverse</u> transformation



T^{-1} expresses s & t as functions of x & y.

<u>Ex</u> $T(r,\theta) = (r\cos\theta,\ r\sin\theta)$, $T^{-1}(x,y) = \left(\sqrt{x^2+y^2},\ \operatorname{arccot}\frac{x}{y}\right)$

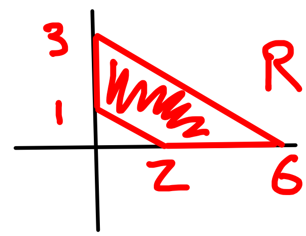
$E = \{(r,\theta) \mid r>0,\ 0<\theta<\pi\}$, $D = \{(x,y) \mid y>0\}$

then $T(E) = D$ & $T^{-1}(D) = E$.

Also, $J(T) = r$ & $J(T^{-1}) = \begin{vmatrix} \frac{x}{(x^2+y^2)^{1/2}} & \frac{y}{(x^2+y^2)^{1/2}} \\ \frac{-1}{1+(x/y)^2}\left(\frac{1}{y}\right) & \frac{-1}{1+(x/y)^2}\left(\frac{-x}{y^2}\right) \end{vmatrix}$

$= \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{r}$.

<u>Ex</u> from last time: trapezoid



diagonal lines are $y = 1-\frac{1}{2}x$, $y = 3-\frac{1}{2}x$, both level sets for $y+\frac{1}{2}x$.

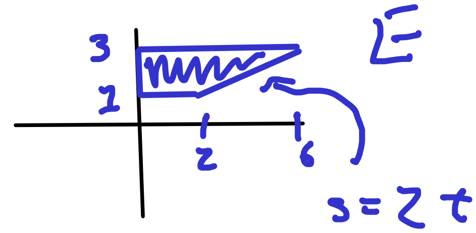
So want $t = y+\frac{1}{2}x$. Keep $s = x$. So $T^{-1}(x,y) = (x,\ y+\frac{1}{2}x)$.

Find T: solve for x & y in terms of s & t.

$x = s$, $y = t-\frac{1}{2}x = t-\frac{1}{2}s$. So $T(s,t) = (s,\ t-\frac{1}{2}s)$.

Then $\iint_R y\,dA = \iint_E \left(t-\frac{1}{2}s\right)|J(T)|\,ds\,dt$

$J(T) = \begin{vmatrix} 1 & 0 \\ -1/2 & 1 \end{vmatrix} = 1$. What is E? $E = T^{-1}(R)$.



$T^{-1}(0,\ 1\le y\le 3) = (0,\ 1\le t\le 3)$

$T^{-1}(0\le x\le 2,\ 1-\frac{1}{2}x)$

$= (0\le s\le 2,\ 1)$

$T^{-1}(0\le x\le 6,\ 3-\frac{1}{2}x) = (0\le s\le 6,\ 3)$

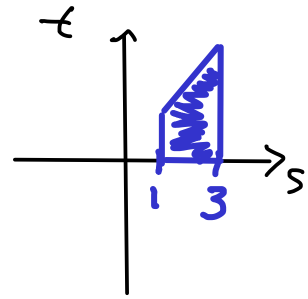
$T^{-1}(2\le x\le 6,\ 0) = (2\le s\le 6,\ 1\le \frac{1}{2}s\le 3)$

So $\iint_R y\,dA = \int_1^3 \int_0^{2t} \left(t - \frac{1}{2}s\right) 1 \, ds\, dt = \dots = \frac{26}{3}$.

<u>Note</u> $J(T^{-1}) = \begin{vmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{vmatrix} = 1 \left(= \frac{1}{1} \right)$

We could have chosen different parametrization, like $T(s,t) = (t, s - \frac{1}{2}t)$

Then $J(T) = \begin{vmatrix} 0 & 1 \\ 1 & -1/2 \end{vmatrix} = -1$.



Sign because orientation got reversed.

Important to remember

absolute value

$$\iint_D f\,dA = \iint_E f\,|J(T)|\,ds\,dt$$

Change of 3 variables

Same procedure, have 3×3 Jacobian matrix

$$T(s,t,u) = \begin{pmatrix} x(s,t,u), \\ y(s,t,u) \\ z(s,t,u) \end{pmatrix} \rightsquigarrow J(T) = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} & \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} & \frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial s} & \frac{\partial z}{\partial t} & \frac{\partial z}{\partial u} \end{vmatrix}$$

And $\iiint_R f(x,y,z)\,dV = \iiint_S f(x(s,t,u), \dots) \left|J(T)\right| ds\,dt\,du$

<u>Ex</u> Cylindrical: $T(r,\theta,z) = (r\cos\theta, r\sin\theta, z)$,

$$J(T) = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r.$$

Ex Spherical: $T(\rho,\phi,\theta) = (\rho\sin\phi\cos\theta,\ \rho\sin\phi\sin\theta,\ \rho\cos\phi)$

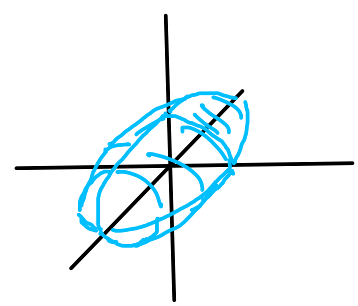
Then
$$J(T) = \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix}$$

$$= \cos\phi\left[\rho^2\cos\phi\sin\phi\cos^2\theta + \rho^2\cos\phi\sin\phi\sin^2\theta\right]$$

$$- -\rho\sin\phi\left[\rho\sin^2\phi\cos^2\theta + \rho\sin^2\phi\sin^2\theta\right]$$

$$= \rho^2\cos^2\phi\sin\phi + \rho^2\sin^2\phi\sin\phi = \boxed{\rho^2\sin\phi}$$

Ex Let R = solid ellipsoid
$$\{(x,y,z) \mid x^2+4y^2+9z^2 \le 1\}$$



Find $\iiint_R 1-x^2-4y^2-9z^2\, dV$

Sphere more convenient, so reparametrize by

$x=s$, $y=\frac{1}{2}t$, $z=\frac{1}{3}u$. So $T(s,t,u) = (s, \frac{1}{2}t, \frac{1}{3}u)$

and $J(T) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{vmatrix} = \frac{1}{6}$.

$$\iiint_R 1-x^2-4y^2-9z^2\, dV = \iiint_{\text{unit sphere}} 1-s^2-t^2-u^2\ \frac{1}{6}\, ds\,dt\,du$$

$$= \frac{1}{6}\int_0^{2\pi}\int_0^{\pi}\int_0^{1} (1-\rho^2)\rho^2\sin\phi\, d\rho\, d\phi\, d\theta = \frac{4\pi}{45}$$

Next time: Green's Theorem.