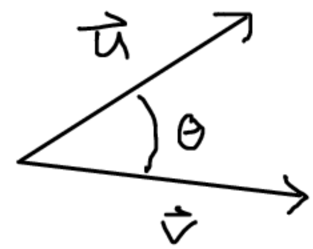
Lecture 3

Aug. 26 2011

Last time: the dot product

$$\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2 + u_3v_3$$

This is a scalar.



Also saw that

$$\vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta,$$

so $\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}$

Examples 1) angle between $\vec{i}$ & $\vec{j}$ is $90°$,

$$\vec{i}\cdot\vec{j} = 0 = 1\cdot 1\cdot\cos 90°$$

We say $\vec{u}$ & $\vec{v}$ are orthogonal (or perpendicular) if $\theta = 90°$. Write $\vec{u}\perp\vec{v}$.

Fact $\vec{u}\perp\vec{v} \iff \vec{u}\cdot\vec{v} = 0.$

means if and only if

2) $(1,1,0)\cdot(2,2,0) = 2+2 = 4$

$\|(1,1,0)\| = \sqrt{2}$, $\|(2,2,0)\| = \sqrt{8} = 2\sqrt{2}$,

$\theta = 0°$ $\qquad 4 = \sqrt{2}\cdot 2\sqrt{2}\cdot\cos 0°$ ✓

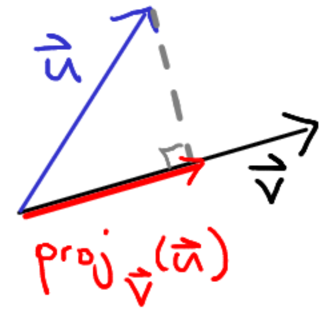
3) $(0,1,0)\cdot(0,1,1) = 1$

$\|(0,1,0)\| = 1$, $\|(0,1,1)\| = \sqrt{2}$, $\theta = 45°$

$1 = 1\cdot\sqrt{2}\cdot\cos 45°$ ✓

Can use dot product to determine orthogonal projections.

Setup: Given vectors $\vec{u}$ & $\vec{v}$, want to determine the part (component) of $\vec{u}$ in the direction of $\vec{v}$.



The vector projection of $\vec{u}$ onto $\vec{v}$ satisfies 1) scalar multiple of $\vec{v}$

2) $(\vec{u} - \text{proj}_{\vec{v}}(\vec{u})) \perp \vec{v}$

Condition 1) $\text{proj}_{\vec{v}}(\vec{u}) = r\vec{v}$, some r

2) $(\vec{u} - r\vec{v}) \cdot \vec{v} = 0$

(Bilinearity) $\vec{u} \cdot \vec{v} = r\, \vec{v} \cdot \vec{v} = r\, \|\vec{v}\|^2$

$$r = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2}$$

$$\boxed{\text{proj}_{\vec{v}}(\vec{u}) = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2}\,\vec{v}}$$

Can also write this as

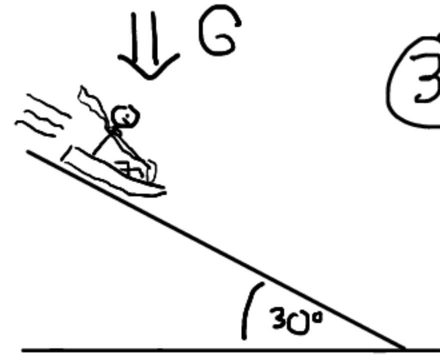
$$\text{proj}_{\vec{v}}(\vec{u}) = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|} \; \frac{\vec{v}}{\|\vec{v}\|}$$

unit vector in direction $\vec{v}$

"scalar projection"

Scalar projection also called component of $\vec{u}$ in direction of $\vec{v}$, written $\text{comp}_{\vec{v}}(\vec{u})$

Note: $\text{comp}_{\vec{v}}(\vec{u}) = \dfrac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|} = \|\vec{u}\| \cos\theta$



Example Jeremy is sledding down a hill of incline 30°. Gravity exerts a force of 9.8 m/s². What is the component of $\vec{G}$ pushing Jeremy down the hill?

$$\vec{G} = (0, -9.8)$$

$$\vec{v} = \left(\tfrac{\sqrt{3}}{2}, -\tfrac{1}{2}\right) \qquad \|\vec{v}\| = 1$$

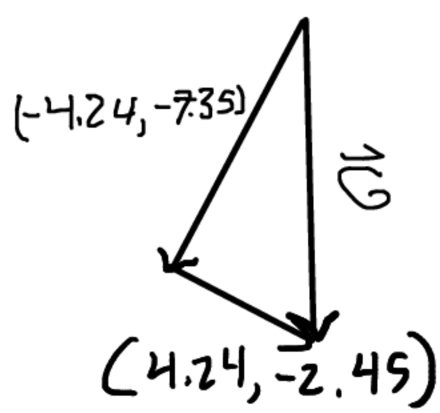
$$\operatorname{comp}_{\vec{v}}(\vec{G}) = \frac{\vec{G} \cdot \vec{v}}{1} = 4.9$$

$$\operatorname{proj}_{\vec{v}}(\vec{G}) = 4.9\,\vec{v} = (2.45\sqrt{3}, -2.45)$$

$$\sim (4.24, -2.45)$$

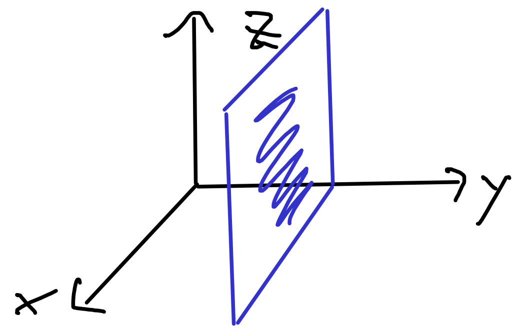
The part of $\vec{G}$ orthogonal to $\vec{v}$ is

$$\vec{G} - 4.9\,\vec{v} \sim (-4.24, -7.35)$$

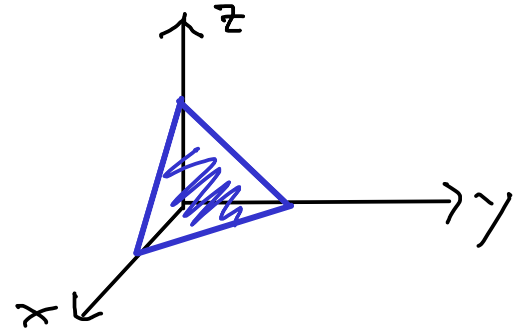


Above: Projection onto a vector, or line (in $\mathbb{R}^2$). In $\mathbb{R}^3$, also study planes.

Ex: 1) $y = z$



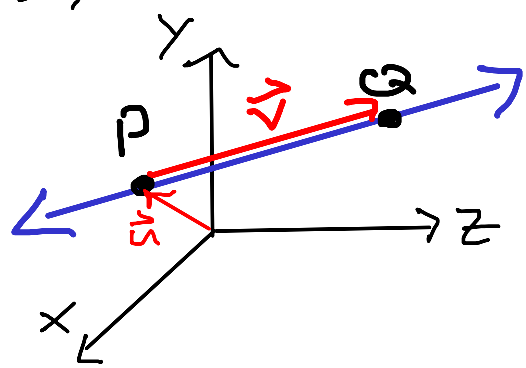
2) $x + y + z = 1$



How to specify lines/planes?

Lines $\mathcal{L}$ Take any P on $\mathcal{L}$.

Take different Q on $\mathcal{L}$.

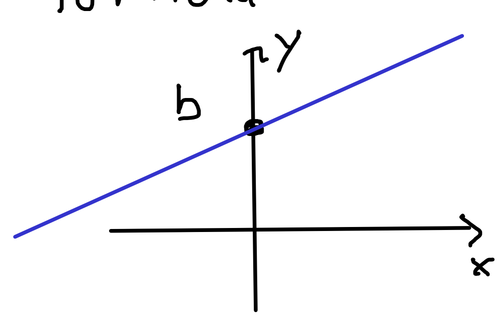


Write $\vec{u} = \overrightarrow{OP}$, $\vec{v} = \overrightarrow{PQ}$.

Then $\mathcal{L} = \{ \vec{u} + t\vec{v} \}$ (parametric form)

Compare to usual $y = mx + b$ formula in $\mathbb{R}^2$. $P = (x, y)$ on $\mathcal{L}$

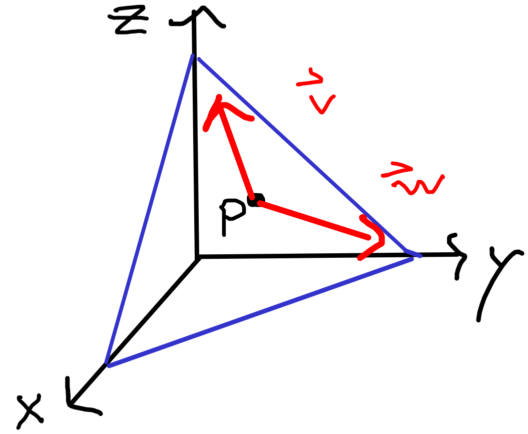
$\Longleftrightarrow$ $P = (0, b) + x(1, m)$



Planes: Try parametric approach.

Plane $\mathcal{P}$, point P

2 degrees of freedom



Again, let $\vec{u} = \overrightarrow{OP}$

For any Q, with $\vec{r} = \overrightarrow{OQ}$, then Q on $\mathcal{P}$

$\iff$ can find s, t so that

$$\vec{r} = \vec{u} + s\vec{v} + t\vec{w}$$

Example $\mathcal{P}$ plane containing $\vec{i}, \vec{j}, \vec{k}$.

Take $P = (1,0,0)$ $(\vec{u} = \vec{i})$.

Choose $\vec{v} = \vec{j} - \vec{i} = (-1, 1, 0)$

$\vec{w} = \vec{k} - \vec{i} = (-1, 0, 1)$.

Is $Q = (\frac{1}{2}, \frac{1}{3}, \frac{1}{6})$ on $\mathcal{P}$?

Want s, t so that

$$\frac{1}{2} = 1 + s(-1) + t(-1)$$

$$\frac{1}{3} = 0 + s(1) + t(0)$$

$$\frac{1}{6} = 0 + s(0) + t(1)$$

$\Rightarrow t = \frac{1}{6}$, $s = \frac{1}{3}$. **YES**