Lecture 30

Nov. 4
2011

Last time Green's theorem $F = (P, Q)$

$$\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \int_{\partial D} F \cdot d\vec{r}$$

Used to replace line int by easier double int
or double int by easier line int

Recall If $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$ on simply conn domain Ω then F is conservative.

This follows from Green's thm:

Enough to show $\int_C F \cdot d\vec{r} = 0$ for every loop in Ω.

Ω simply conn $\Rightarrow$ C is boundary of some region D in Ω

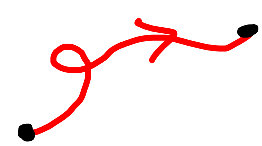
Then $\int_C F \cdot d\vec{r} = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = 0$ //

Next few lectures: Surfaces & Surface Integrals

§16.6 Parametric surfaces.

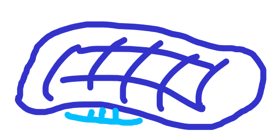
Parametric curves: $\vec{r} : [a,b] \longrightarrow \mathbb{R}^3$

$C = \vec{r}([a,b])$

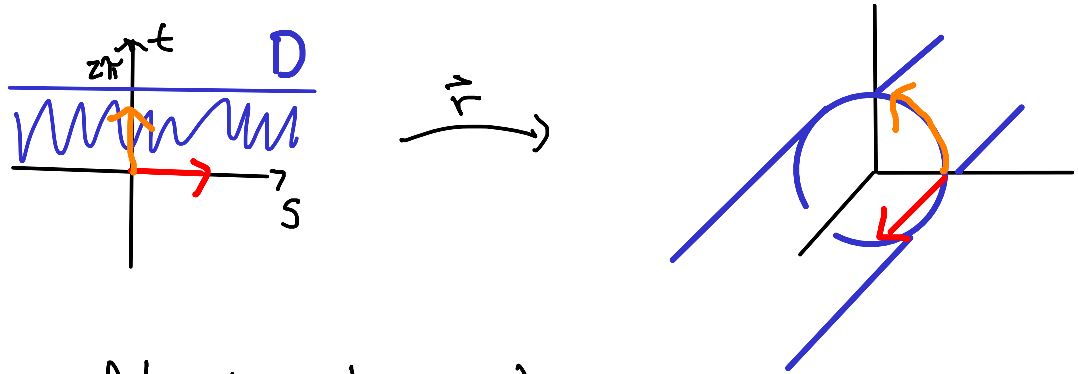


Parametric surfaces: $\vec{r} : D \longrightarrow \mathbb{R}^3$

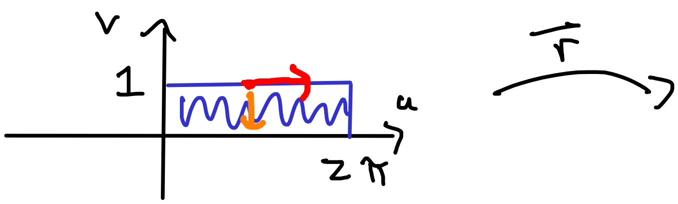
some region in $\mathbb{R}^2$

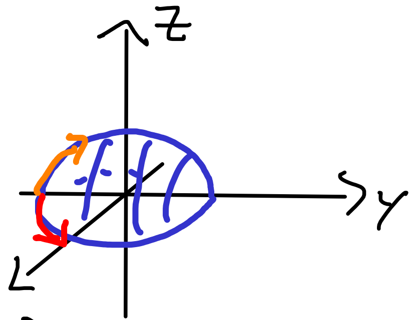


Ex Cylinder: $\vec{r}(s,t) = (s, \cos t, \sin t)$ $\quad y^2+z^2=1$



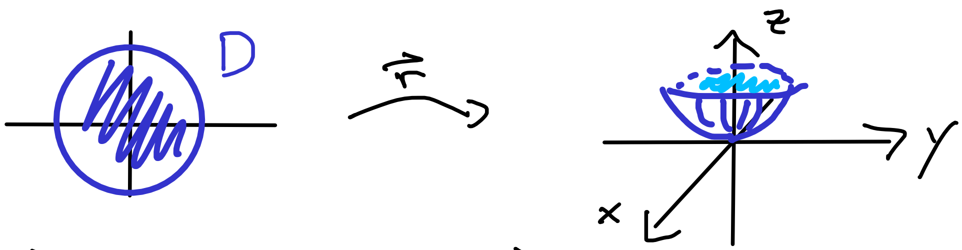
Ex Northern hemisphere





$$\vec{r}(u,v) = (v\cos u,\ v\sin u,\ \sqrt{1-v^2})$$

Ex Any $f:\mathbb{R}^2 \to \mathbb{R}$, e.g. $f(x,y) = x^2+y^2$

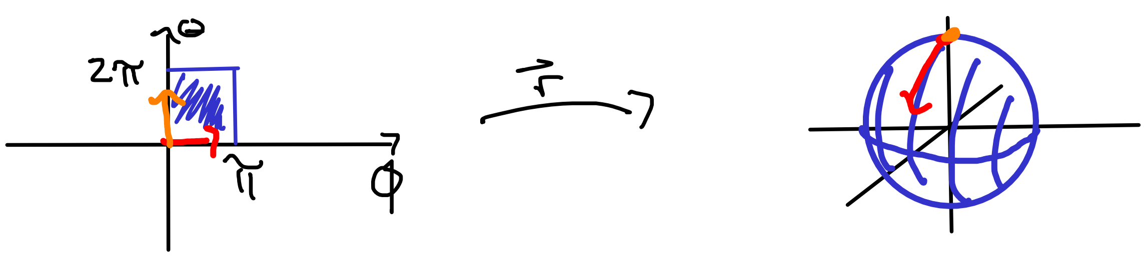


$\vec{r}(u,v) = (u,v,f(u,v))$ parametrizes graph of f.

$\vec{r}(u,v) = (u,v,\sqrt{1-u^2-v^2})$ gives another parametrization of northern hemisphere.

One more parametrization of entire unit sphere comes from spherical coords:

$$\vec{r}(\phi,\theta) = (\sin\phi\cos\theta,\ \sin\phi\sin\theta,\ \cos\phi)$$



How to find tangent planes

Recall tangent plane to level surface $f(x,y,z)=0$ at (a,b,c) has normal vector $\nabla f(a,b,c)$, so plane described by $\nabla f(a,b,c)\cdot(x-a,\ y-b,\ z-c)=0.$

Ex Sphere given by $\underbrace{x^2+y^2+z^2}_{f}=1$

$$\nabla f=(2x,2y,2z) = \text{scalar mult of } (x,y,z)$$

Plane given by $a(x-a)+b(y-b)+c(z-c)=0$

or $ax+by+cz=a^2+b^2+c^2=1.$

What about tangent plane to parametrized surface?

$f\circ\vec{r}=$ constant, so

$$\frac{\partial}{\partial u}(f\circ\vec{r})=\frac{\partial}{\partial v}(f\circ\vec{r})=0.$$

Chain rule: $\frac{\partial}{\partial u}(f\circ\vec{r})=\frac{\partial f}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial u}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial u}$

$$=\nabla f\cdot\frac{\partial\vec{r}}{\partial u}$$

$\frac{\partial\vec{r}}{\partial u}\cdot\nabla f=0$, so $\frac{\partial\vec{r}}{\partial u}$ lies on tangent plane.

Same for $\frac{\partial\vec{r}}{\partial v}$.

The vector $\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}$ gives normal vector for tangent plane.

Ex Back to sphere.

Using $\vec{r}(u,v)=(v\cos u,\ v\sin u,\ \sqrt{1-v^2})$,

$$\frac{\partial \vec{r}}{\partial u} = (-v\sin u,\ v\cos u,\ 0)$$

$$\frac{\partial \vec{r}}{\partial v} = \left(\cos u,\ \sin u,\ \frac{-v}{\sqrt{1-v^2}}\right)$$

normal vect

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -v\sin u & v\cos u & 0 \\ \cos u & \sin u & \frac{-v}{\sqrt{1-v^2}} \end{vmatrix} = \frac{-v^2\cos u}{\sqrt{1-v^2}}\vec{i} - \frac{v^2\sin u}{\sqrt{1-v^2}}\vec{j}$$

$$+ (-v\sin^2 u - v\cos^2 u)\vec{k}$$

$$= \left(\frac{-v^2\cos u}{\sqrt{1-v^2}},\ \frac{-v^2\sin u}{\sqrt{1-v^2}},\ -v\right)$$

$$= \frac{-v}{\sqrt{1-v^2}}\left(v\cos u,\ v\sin u,\ \sqrt{1-v^2}\right)$$

Using $\vec{r}(\phi,\theta) = (\sin\phi\cos\theta,\ \sin\phi\sin\theta,\ \cos\phi)$

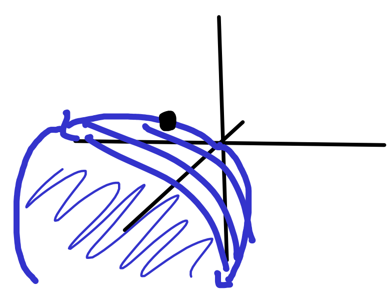
find $\frac{\partial \vec{r}}{\partial \phi} \times \frac{\partial \vec{r}}{\partial \theta} = \sin\phi\, \vec{r}(\phi,\theta)$.

Ex Find tangent plane to surface param by

$\vec{r}(u,v) = (u^2,\ u\sin v,\ u\cos v)$ at $\vec{r}(1,0)$

$= (1,0,1)$

$$\frac{\partial \vec{r}}{\partial u} = (2u,\ \sin v,\ \cos v)$$



$$\frac{\partial \vec{r}}{\partial v} = (0,\ u\cos v,\ -u\sin v)$$

$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2u & \sin v & \cos v \\ 0 & u\cos v & -u\sin v \end{vmatrix} = \left(-u(\sin^2 v + \cos^2 v),\ 2u^2\sin v,\ 2u^2\cos v\right)$$

$$= u(-1,\ 2u\sin v,\ 2u\cos v)$$

@ $u=1, v=0$ get $\vec{n} = (-1, 0, 2)$

Tangent plane: $\vec{n}\cdot(x-1, y, z-1) = 0$ or $-x + 2z = 1$

Note: $\frac{\partial \vec{r}}{\partial v}(0,v) = \vec{0}$ so can't see tangent plane at $\vec{r}(0,v) = (0,0,0)$.

The surface is the elliptic paraboloid $x = y^2 + z^2$.
Under parametrization $\vec{r}(u,v) = (u^2+v^2, u, v)$

$$\frac{\partial \vec{r}}{\partial u} = (2u, 1, 0), \quad \frac{\partial \vec{r}}{\partial v} = (2v, 0, 1)$$

& $$\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = (1, -2u, -2v)$$

So tangent plane at (a,b,c) given by

$$(1,-2b,-2c)\cdot(x-a, y-b, z-c) = 0$$

or $$x - 2by - 2cz = a - 2b^2 - 2c^2.$$

Tangent plane @ $(0,0,0)$ given by $x=0$.
@ $(1,0,1)$ given by $x - 2z = -1$.

Next time: surface area, surface integrals.