Lecture 31

Nov. 7 2011 ①

Quiz Thursday: Change of coords, Green's Thm, section 16.6

Last time: parametrizing surfaces $\vec{r}(u,v)$

Tangent plane @ $\vec{r}(u,v)$ contains vectors

$\frac{\partial \vec{r}}{\partial u}(u,v)$, $\frac{\partial \vec{r}}{\partial v}(u,v)$ $\rightsquigarrow$ normal vector $\frac{\partial \vec{r}}{\partial u}(u,v) \times \frac{\partial \vec{r}}{\partial v}(u,v)$.

Ex S = graph of $g(x,y)$

get "obvious" parametrization $\vec{r}(u,v) = (u, v, g(u,v))$

Then $\vec{r}_u = (1, 0, g_u)$, $\vec{r}_v = (0, 1, g_v)$,

$\vec{r}_u \times \vec{r}_v = -g_u \vec{i} - (g_v)\vec{j} + \vec{k} = (-g_u, -g_v, 1)$.

So tangent plane at $(a,b,c) = (a, b, g(a,b))$ given by

$$-g_u(a,b)(x-a) - g_v(a,b)(y-b) + z - c = 0.$$

$z = c + g_u(a,b)(x-a) + g_v(a,b)(y-b)$ formula from §14.4

Note: not every surface has well-defined tangent planes at all points!

Ex Cone $z = \sqrt{x^2+y^2} = g(x,y)$.

$$g_x = \frac{x}{\sqrt{x^2+y^2}}, \quad g_y = \frac{y}{\sqrt{x^2+y^2}}$$

Tangent plane given at $(a, b, g(a,b))$ by

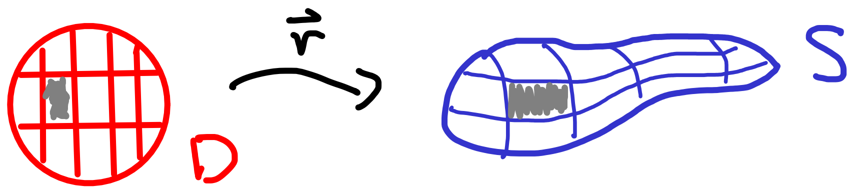
$$z = \sqrt{a^2+b^2} + \frac{a}{\sqrt{a^2+b^2}}(x-a) + \frac{b}{\sqrt{a^2+b^2}}(y-b)$$

$$= \frac{ax}{\sqrt{a^2+b^2}} + \frac{by}{\sqrt{a^2+b^2}} \qquad \text{for } (a,b) \neq (0,0)$$

g_u, g_v not defined at $(0,0)$.
No tangent plane.

Today & Wed: surface area, surface integrals

Divide surface into small pieces coming from param $\vec{r}$.



Area $= \Delta u \Delta v$ in D.

In S, approximate region by parallelogram with sides of length $\Delta u \frac{\partial \vec{r}}{\partial u}$ & $\Delta v \frac{\partial \vec{r}}{\partial v}$.

Then area $= \Delta u \Delta v \cdot \left\| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right\|$.

Add these up using small partitions of D

$\rightsquigarrow$ Surface area $= \iint_D \| \vec{r}_u \times \vec{r}_v \| \, dA$.

Ex The sphere (of course) radius $= d$.

Last time, used $\vec{r}(u,v) = (v\cos u, v\sin u, \sqrt{d^2-v^2})$

$\vec{r}_u = (-v\sin u, v\cos u, 0)$

$\vec{r}_v = \left(\cos u, \sin u, \frac{-v}{\sqrt{d^2-v^2}}\right)$

Parametrizes northern hemisphere

Found $\vec{r}_u \times \vec{r}_v = \frac{-v}{\sqrt{d^2-v^2}} \vec{r}$

So $\| \vec{r}_u \times \vec{r}_v \| = \frac{|v|}{\sqrt{d^2-v^2}} \cdot d$

Then area(hemisphere) $= \iint \frac{|v|}{\sqrt{d^2-v^2}}\, d\, dA$

$[0,2\pi]\times[0,d] \longrightarrow D$

$$= d\int_0^d \int_0^{2\pi} \frac{|v|}{\sqrt{d^2-v^2}}\, du\, dv$$

$$= 2\pi d \int_0^d \frac{v}{\sqrt{d^2-v^2}}\, dv \qquad \begin{aligned} x &= d^2-v^2 \\ dx &= -2v\, dv \end{aligned}$$

$$= \frac{-1}{2} 2\pi d \int_{d^2}^0 \frac{1}{\sqrt{x}}\, dx = -\pi d \; 2\sqrt{x}\,\Big|_{d^2}^0 = 2\pi d^2$$

So area of sphere $= 4\pi d^2$.

Using spherical param $r(\phi,\theta) = (d\sin\phi\cos\theta, d\sin\phi\sin\theta, d\cos\phi)$

get $\|\vec{r}_u \times \vec{r}_v\| = d|\sin\phi|\;\|\vec{r}\| = d^2\sin\phi$

Area $= \iint_D d^2\sin\phi\, dA = \int_0^\pi\int_0^{2\pi} d^2\sin\phi\, d\theta\, d\phi$

$$= 2\pi d^2 \int_0^\pi \sin\phi\, d\phi = -2\pi d^2\cos\phi\,\Big|_0^\pi = 4\pi d^2$$

What does this say in case $S = \text{graph}(g)$?

Saw $\vec{r}_u \times \vec{r}_v = (-g_u, -g_v, 1)$, so

$$\text{Area}(\text{graph}(g)) = \iint_D \sqrt{g_u^2 + g_v^2 + 1}\, dA$$

alternative notation

$$= \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

(like $\int \sqrt{1+(f'(t))^2}\, dt$ formula for arclength)

§ 16.7 Surface Integrals

Review line integrals of functions $f: \mathbb{R}^3 \to \mathbb{R}$.

Started with $\text{arclength}(C) = \int_C ds$.

Then introduced $\int_C f\, ds$ as means of computing average value of f along C.

$$\begin{matrix}\text{average value} \\ \text{of } f \text{ on } C\end{matrix} = \frac{1}{\text{length}(C)} \int_C f\, ds.$$

Same idea for surface integrals.

S surface parametrized by $\vec{r}(u,v)$, $f: \mathbb{R}^3 \to \mathbb{R}$ function. Define surface integral

$$\iint_S f\, \underbrace{dS}_{\text{surface area parameter}} = \iint_D f \circ \vec{r}\, \|\vec{r}_u \times \vec{r}_v\|\, dA$$

$$\begin{matrix}\text{Average value} \\ \text{of } f \text{ on } S\end{matrix} = \frac{1}{\text{area}(S)} \iint_S f\, dS$$

$\iint_S dS$

Ex Find average value of $f(x,y,z) = xz + y^2$ on portion of cone $z^2 = x^2 + y^2$ inside cylinder $x^2 + y^2 = 4$.

Already saw one parametrization, but the param

$\vec{r}(u,v) = (u\cos v, u\sin v, u)$ more convenient.

$D = [0,2] \times [0, 2\pi]$

$$\vec{r}_u = (\cos v,\ \sin v,\ 1)$$
$$\vec{r}_v = (-u\sin v,\ u\cos v,\ 0) \rightsquigarrow \vec{r}_u \times \vec{r}_v = (-u\cos v, -u\sin v, u)$$ ⑤

$$\|\vec{r}_u \times \vec{r}_v\| = \sqrt{2u^2} = u\sqrt{2}.$$

$$\text{Area} = \iint_D \|\vec{r}_u \times \vec{r}_v\|\, dA = \int_0^2\int_0^{2\pi} u\sqrt{2}\ dv\,du$$
$$= 2\sqrt{2}\pi \int_0^2 u\,du = 4\sqrt{2}\pi.$$

$$\text{Average value} = \frac{1}{4\sqrt{2}\pi}\int_0^2\int_0^{2\pi} (u^2\cos v + u^2\sin^2 v)\, u\sqrt{2}\, dv\,du$$
$$= \frac{1}{4\pi}\int_0^2 u^3\,du \underbrace{\int_0^{2\pi} \cos v + \sin^2 v\, dv}_{\hookrightarrow\ \frac{v}{2} - \frac{\sin 2v}{4}\Big|_0^{2\pi}}$$
$$= \frac{1}{4\pi}\cdot 4\cdot \pi = 1$$