# Lecture 32

Nov. 9 2011 ①

Quiz Thursday: Change of coords, Green's Thm, section 16.6

Last time: Surface integrals

$$\iint_S f\,dS = \iint_D f\circ\vec{r}\,\|\vec{r}_u\times\vec{r}_v\|\,dA,$$

$\vec{r}: D\to\mathbb{R}^3$ parametrizes $S$.

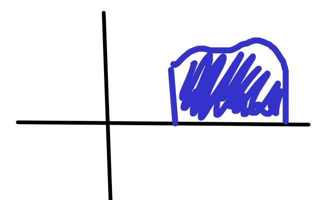
Used for measuring average value of $f$ on $S$.

$$\text{average val of } f \text{ on } S = \frac{1}{\text{surface area}(S)}\iint_S f\,dS$$

What does this measure?

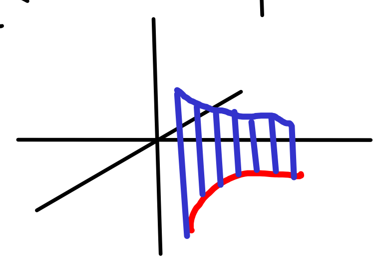
Back to 1-var integrals: $\int_a^b f(x)\,dx$

computes area of region in $\mathbb{R}^2$



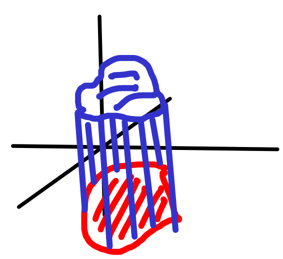
line integral $\int_C f\,ds$

compute area of region in $\mathbb{R}^3$



double integral: $\iint_D f\,dA$

Compute volume of region in $\mathbb{R}^3$



surface integral: $\iint_S f\,dS$

compute volume of region in $\mathbb{R}^4$ can't draw.

Problem: graph of $f: S\to\mathbb{R}$ contained in $\mathbb{R}^4$.

Warning: Don't try to just visualize the solid in $\mathbb{R}^3$

Ex $S$ = unit sphere (has area $4\pi$),

$f$ = constant function $2$.

In $\mathbb{R}^3$, might guess solid is sphere of radius 3 w/ unit sphere removed. This would have volume $4/3\,\pi(3)^3 - 4/3\,\pi\cdot(1)^3 = 4/3\,\pi(26) = \frac{104}{3}\pi$

$\approx 109$

But $\iint_S 2\,dS = 2$ surface area $= 8\pi \approx 25$

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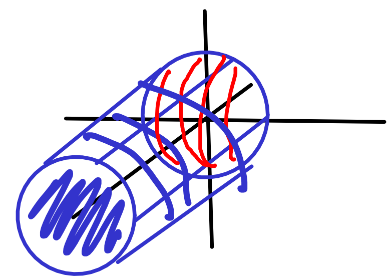
Ex Surface $S$ has sides cylinder $y^2+z^2=1$, $x = 1-y^2-z^2$, $x=4$ (pop can).



How to set up $\iint_S x^2+yz$ ?

Use 3 integrals. $\underset{S_1}{x=4}$ end use param $\vec{r}(u,v)=(4, v\cos u, v\sin u)$

$$\iint_{S_1} x^2+yz\,dS = \int_0^{2\pi}\int_0^1 [16 + v^2\cos u\sin u]\,\|\vec{r}_u\times\vec{r}_v\|\,dv\,du$$

$\vec{r}_u = (0, -v\sin u, v\cos u)$

$\vec{r}_v = (0, \cos u, \sin u)$

$\|(-v, 0, 0)\|$

$= |v| = v$

$S_2$ cylinder use param $\vec{r}(u,v) = (v, \cos u, \sin u)$

$\vec{r}_u = (0, -\sin u, \cos u)$

$\vec{r}_v = (1, 0, 0)$

$\|\vec{r}_u\times\vec{r}_v\|$

$= \|(0, \cos u, \sin u)\| = 1$

$$\iint_{S_2} x^2+yz\,dS = \int_0^{2\pi}\int_0^4 v^2 + \cos u\sin u\,dv\,du$$

$S_3$ part, use param $\vec{r}(u,v) = (1-v^2, v\cos u, v\sin u)$

$$\left.\begin{matrix}\vec{r}_u = (0, -v\sin u, v\cos u \\ \vec{r}_v = (-2v, \cos u, \sin u\end{matrix}\right)$$

$$\|\vec{r}_u \times \vec{r}_v\| = \|(-v, -2v^2\cos u, -2v^2\sin u)\| = \sqrt{v^2+4v^4} = v\sqrt{1+2v^2}$$

$$\iint_{S_3} x^2 + yz\,dS = \int_0^{2\pi}\int_0^1 \left[(1-v^2)^2 + v^2\cos u\sin u\right] v\sqrt{1+2v^2}\,dv\,du.$$

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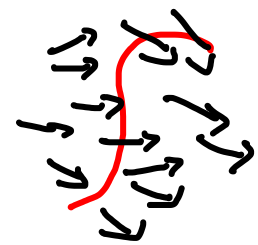
Exam 3 covers up to this point.

Next Surface integrals of vector fields. Will see has to do with "flux".

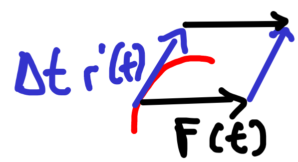
Rate of flow (of fluid, gas, ...)

First study flux across curves.

Flow given by velocity vector field F.

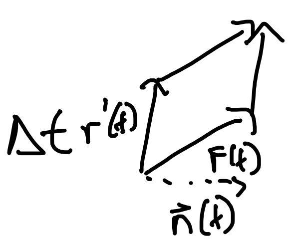


Look at small piece of curve C, param by $\vec{r}(t)$. Amount flowing




through over this portion of curve approximated by area of parallelogram w/ sides $\Delta t\, r'(t)$ & $F(t)$.

Area $= |F(t) \times r'(t)|\,\Delta(t)$. Not the most convenient formulation.




Take $\vec{n}(t)$ unit normal vector to C (orthog to $r'(t)$).

Then Area = base · height $= \Delta t\,\|r'(t)\|\,(\vec{F}(t)\cdot\vec{n}(t))$

Now sum up over $t$, get

$$\text{Flux} = \int_{t_0}^{t_1} F(t) \cdot \vec{n}(t) \, \|\vec{r}\,'(t)\| \, dt$$

$$= \int_C F(t) \cdot \vec{n}(t) \, ds$$

<u>Note</u>: 2 possible directions for $\vec{n}(t)$.

This choice dictates in which direction we measure flux.