Lecture 33

Nov. 11
2011
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Exam rooms for Ex3 same as for Ex2.

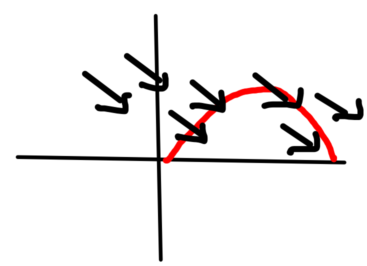
Last time: "Flux" = rate of flow through curve C.

$\vec{F}$ = velocity vector field for flow,

$\vec{r}(t)$ parametrization of C, $\vec{n}(t)$ unit normal vector to C at $\vec{r}(t)$.

Then $\text{Flux} = \int_C \vec{F}\cdot\vec{n}\, ds$

Ex $F(x,y) = (1,-1)$ constant



C = sine curve between $x=0$, $x=\pi$

$\vec{r}(t) = (t, \sin t) \quad 0 \le t \le \pi.$

$\vec{r}\,'(t) = (1, \cos t) \quad \|\vec{r}\,'(t)\| = \sqrt{1+\cos^2 t}$

$\vec{n}(t) = \dfrac{(\cos t, -1)}{\sqrt{1+\cos^2 t}}$

$$\int_C F\cdot\vec{n}\, ds = \int_0^{\pi} \frac{\cos t + 1}{\sqrt{1+\cos^2 t}} \underbrace{\sqrt{1+\cos^2 t}}_{\|r'(t)\|}\, dt$$

$$= \int_0^{\pi} \cos t + 1\, dt = \pi.$$

Note: we chose $\vec{n}(t) = \dfrac{(\cos t, -1)}{\sqrt{1+\cos^2 t}}$. Could have chosen unit normal $\vec{n}(t) = \dfrac{(-\cos t, 1)}{\sqrt{1+\cos^2 t}}$. Then get flux $= -\pi$.

Choice of $\vec{n} \longleftrightarrow$ choice of "orientation".

Note: If $\vec{r}\,'(t) = (x'(t), y'(t))$

then $(y'(t), -x'(t))$ is a normal vector

& $\frac{1}{\sqrt{y'(t)^2 + x'(t)^2}}(y'(t), -x'(t))$ unit normal vect.

But $\|\vec{r}\,'(t)\| = \sqrt{x'(t)^2 + y'(t)^2}$, so

$$\int_C F \cdot \vec{n}\, ds = \int_{t_0}^{t_1} F(\vec{r}(t)) \cdot \frac{(y'(t), -x'(t))}{\cancel{\sqrt{x'(t)^2 + y'(t)^2}}} \cancel{\sqrt{x'(t)^2 + y'(t)^2}}\, dt$$

$$= \int_{t_0}^{t_1} F(\vec{r}(t)) \cdot (y'(t), -x'(t))\, dt.$$

So no need to even calculate $\|\vec{r}\,'(t)\|$.

Writing $\vec{F} = (P, Q)$, get

$$\int_C F \cdot \vec{n}\, ds = \int_C P\, y'(t) - Q\, x'(t)\, dt$$

$$= \int_C (-Q, P) \cdot r'(t)\, dt$$

In case of arc length parametrization, so $\|r'(t)\| = 1$,

get $\int_C \vec{F} \cdot \vec{n}\, ds = \int_C (-Q, P) \cdot d\vec{r}$

If $C = \partial D$ some region D, get (by Green's thm)

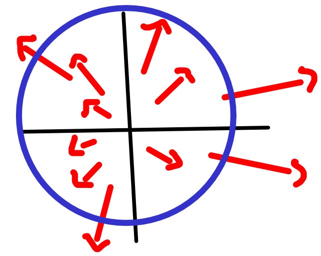
$$\int_C \vec{F} \cdot \vec{n}\, ds = \iint_D \frac{\partial}{\partial x} P - \frac{\partial}{\partial y}(-Q)\, dA$$

$$= \iint_D \underbrace{\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}}_{\text{The "divergence" of } F.}\, dA.$$

$$\operatorname{div}(F) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}.$$ May write $\operatorname{div}(F) = \nabla \cdot F$

Measures how much flow $\vec{F}$ "expands" or "diverges" at point.
(Rate of change of flow)

<u>Ex</u> $F(x,y) = (x,y)$



C = unit circle.

Have arclength param $\vec{r}(t) = (\cos t, \sin t)$.

$\vec{r}\,'(t) = (-\sin t, \cos t)$ $\quad \vec{n}(t) = (\cos t, \sin t)$

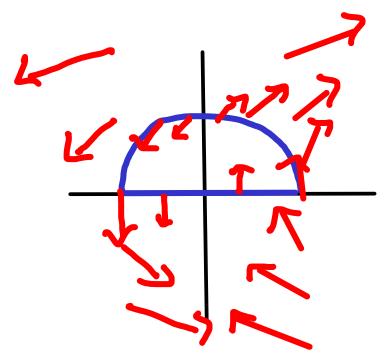
$(0,0)$ is a "source"

$$\text{Flux} = \int_C F \cdot \vec{n}\, ds = \int_0^{2\pi} (\cos t, \sin t)\cdot(\cos t, \sin t)\, dt = 2\pi$$

$$\iint_D \operatorname{div}(F)\, dA = \iint_D 2\, dA = 2 \cdot \text{area} = 2\pi.$$

The statement $\underbrace{\int_{\partial D} F \cdot \vec{n}\, ds}_{\text{Flux}} = \iint_D \operatorname{div}(\vec{F})\, dA$

is called the <u>Divergence Theorem</u>



<u>Ex</u> $F(x,y) = (xy, x)$

Then flux F across C

$$= \int_C F \cdot \vec{n}\, ds = \iint_D \operatorname{div} F\, dA$$

Calculate line integral: $\int_C F\cdot\vec{n}\, ds = \int_{C_1} F \cdot \vec{n}\, ds + \int_{C_2} F\cdot\vec{n}\, ds$

C_1 param by $(\cos t, \sin t)$ $\quad 0 \le t \le \pi$

$\vec{n}(t) = (\cos t, \sin t)$

$$\int_{C_1} F \cdot \vec{n}\, ds = \int_0^\pi (\cos t \sin t, \cos t) \cdot (\cos t, \sin t)\, dt$$

$$= \int_0^\pi \cos^2 t \sin t + \cos t \sin t \, dt$$

$$= \frac{-\cos^3 t}{3} + \frac{\sin^2 t}{2} \Big|_0^\pi = -2/3$$

C_2: $\vec{r}(t) = (t, 0) \quad -1 \le t \le 1 \quad \vec{n}(t) = (0, -1)$

$$\int_{C_2} F \cdot \vec{n}\, ds = \int_{-1}^1 (0, t) \cdot (0, -1)\, dt = \int_{-1}^1 -t\, dt = 0.$$

So

$$\iint_D y\, dA = \int_0^\pi \int_0^1 r^2 \sin\theta \, dr\, d\theta$$

$$= \frac{1}{3} \cdot (2) = 2/3$$

Next time Flux for surfaces