Lecture 34

Nov. 14 2011

No lecture on Friday.

Last time: "Flux" = rate of flow through curve C.

$\vec{F}$ = velocity vector field for flow,

$\vec{r}(t)$ parametrization of C, $\vec{n}(t)$ unit normal vector to C at $\vec{r}(t)$.

Then $\text{Flux} = \int_C \vec{F} \cdot \vec{n}\, ds$

Divergence Theorem

When $C = \partial D$, then $\int_C \vec{F} \cdot \vec{n}\, ds = \iint_D \text{div}(\vec{F})\, dA$,

where $\text{div}(\vec{F}) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$.

In order for signs to work out, need

1) C oriented counterclockwise (C for Green)

2) If $\vec{r}(t) = (x(t), y(t))$, pick $\vec{n}(t) = \underbrace{(y'(t), -x'(t))}_{\|\ \ \ \|}$

This is the outward-pointing normal vector.

Ex Think about unit circle w/ $\vec{r}(t) = (\cos t, \sin t)$.

Then $\vec{n}(t) = \left(\frac{d}{dt}\sin t, -\frac{d}{dt}\cos t\right) = (\cos t, \sin t)$ points outward.

The signs work out the same if we instead take

1′) C oriented clockwise

2′) If $\vec{r}(t) = (x(t), y(t))$, $\vec{n}(t) = \underbrace{(-y'(t), x'(t))}_{\|\ \ \ \|}$

In this case, $\vec{n}$ also pointing outward.

Ex unit circle w/ $\vec{r}(t) = (\cos t, -\sin t)$

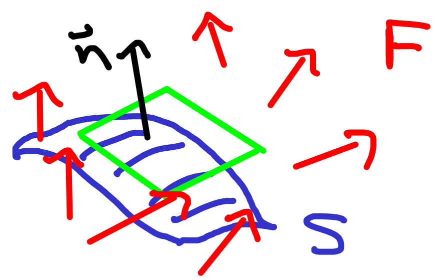
Then $\vec{n}(t) = \left(-\frac{d}{dt}(-\sin t), \frac{d}{dt}\cos t\right)$

$= (\cos t, -\sin t)$

points outward.

Moral of the story: In the divergence theorem, want outward pointing normal vector.

Today: Flux through Surfaces.



Same idea as for curves:

Take param $\vec{r}(u,v)$ of S.

To measure flux near point $\vec{r}(u,v)$,

approximate S by parallelogram with sides $(\Delta u)\vec{r}_u$ & $(\Delta v)\vec{r}_v$

Measure volume of prism w/ 3rd side given by $F(\vec{r}(u,v))$.

Formula: $\text{Volume} = \underbrace{\text{area(base)}}_{\Delta u \Delta v \|r_u \times r_v\|} \cdot \underbrace{\text{height}}_{F(\vec{r}(u,v)) \cdot \vec{n}(u,v)}$

$\vec{n} = \dfrac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}$ unit normal vector.

So $\text{Volume} = \left[F(\vec{r}) \cdot \vec{r}_u \times \vec{r}_v\right] \Delta u \Delta v$

Add up $\rightsquigarrow$ $\text{Flux} = \iint_S \vec{F} \cdot \vec{n} \, dS$

$= \iint_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) \, dA$

(Also written $\iint_S \vec{F} \cdot d\vec{S}$)

Ex S = paraboloid $z = 1 - x^2 - y^2$, $0 \le z \le 1$

$$F = (y, x, 1)$$

$$\vec{r}(u,v) = (u\cos v,\ u\sin v,\ 1-u^2)$$

$$D = [0,1] \times [0, 2\pi].$$

$$\vec{r}_u = (\cos v,\ \sin v,\ -2u) \qquad \vec{r}_u \times \vec{r}_v = (2u^2\cos v,\ 2u^2\sin v,\ u)$$

$$\vec{r}_v = (-u\sin v,\ u\cos v,\ 0)$$

Then $F(\vec{r}) \cdot (\vec{r}_u \times \vec{r}_v) = (u\sin v, u\cos v, 1) \cdot$

$$= 2u^3 \cos v \sin v + 2u^3 \cos v \sin v + u$$

$$= 2u^3 \sin 2v + u$$

$$\text{Flux} = \iint_S F \cdot \vec{n}\, dS = \int_0^{2\pi}\int_0^1 2u^3 \sin 2v + u \, du\, dv$$

$$= \int_0^{2\pi} \tfrac{1}{2} \sin 2v + \tfrac{1}{2}\, dv = \pi$$

<u>Alternative</u> Think of S as graph of $g(x,y) = 1 - x^2 - y^2$,

use param $\vec{r}(u,v) = (u, v, g(u,v))$.

$$\vec{r}_u = (1, 0, g_u) = (1, 0, -2u) \qquad \vec{r}_u \times \vec{r}_v = (-g_u, -g_v, 1)$$

$$\vec{r}_v = (0, 1, g_v) = (0, 1, -2v) \qquad = (2u, 2v, 1)$$

$$F(\vec{r}) \cdot (\vec{r}_u \times \vec{r}_v) = F_3 - g_u F_1 - g_v F_2$$

$$= \underline{1} + 2uv + 2uv = 1 + 4uv$$

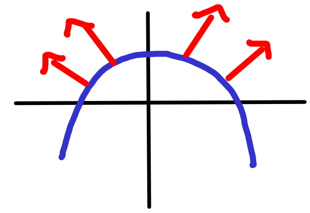
$$\text{Flux} = \iint_S F \cdot \vec{n}\, dS = \iint_D 1 + 4uv \, dA$$

$$= \int_0^{2\pi}\int_0^1 [1 + 4r^2 \cos\theta \sin\theta]\, r\, dr\, d\theta$$

$$= \dots = \pi$$

We chose $\vec{n} = \dfrac{\vec{r}_u \times \vec{r}_v}{\| \quad \|}$.

In $x=0$ slice, looks like



$\vec{r}_u \times \vec{r}_v = (-g_u, -g_v, 1)$, so

$\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{\| \quad \|}$ always has <u>positive</u> z component.

With this choice, $\iint_S F \cdot dS = \iint_D F_3 - g_u F_1 - g_v F_2 \, dA$

Note that $\vec{r}_v \times \vec{r}_u = (g_u, g_v, -1)$ gives another choice w/ unit normal having <u>negative</u> z component.

With this choice, get $\iint_S F \cdot dS = \iint_D g_u F_1 + g_v F_2 - F_3 \, dA$.

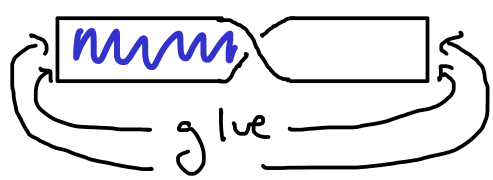
There are <u>usually</u> two choices for n. Why not always?

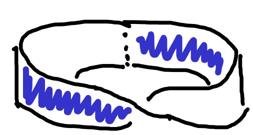
Want continuously defined unit normal $\vec{n}(t)$.

Does not always exist:

<u>Ex</u> S = Moëbius band M

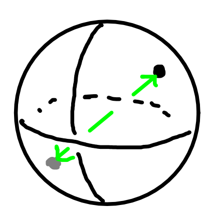
This surface only has one "side". If $\vec{n}$ starts pointing "outwards", travel once around band. Then $\vec{n}$ must point "inwards".





M

<u>Ex</u> S = projective plane $\mathbb{RP}^2$



Start with sphere.

Glue each point to polar opposite.

Get same result by starting w/ hemisphere and gluing points on boundary circle. Gluing identifies "outward" $\vec{n}$ with "inward" $\vec{n}$, so no well-defined outward $\vec{n}$ for $\mathbb{RP}^2$.

We say S is orientable if there is continuous choice of unit normal vector $\vec{n}$. If S is orientable, there are two choices for $\vec{n}$. (When computing Flux, usually pick outward choice.)