Lecture 35

$\boxed{\text{No Lecture Friday}}$

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Previously Flux across curve C,
divergence theorem $\int_C F \cdot \vec{n}\, ds = \iint_D \operatorname{div} F \, dA$.

Flux across surface S $\iint_S F \cdot \vec{n}\, dS$.

Today Divergence theorem for surfaces.

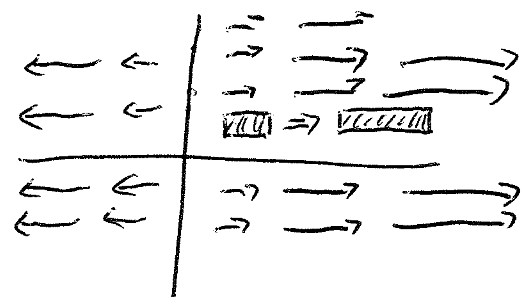
$F : \mathbb{R}^3 \to \mathbb{R}^3$, divergence is

$$\operatorname{div}(\vec{F}) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \text{ "=" } \nabla \cdot \vec{F}$$

~~Divergence Theorem~~

Divergence measures compression/expansion of fluid/gas.

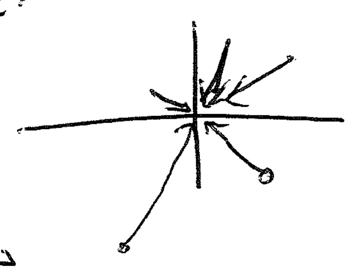
Ex $\vec{F}(x,y) = (x,0) = x\vec{i}$



Gas expanding

$$\operatorname{div} \vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(0) = 1 > 0.$$

Ex $\vec{F}(x,y) = (-x,-y)$



Liquid being compressed towards $(0,0)$. Expect $\operatorname{div} \vec{F} < 0$.

$$\operatorname{div} \vec{F} = \frac{\partial}{\partial x}(-x) + \frac{\partial}{\partial y}(-y) = -1 + -1 = -2 < 0.$$

Flow lines: curve C param by $\vec{r}(t)$, so that $\vec{r}\,'(t) = \vec{F}(\vec{r}(t))$.

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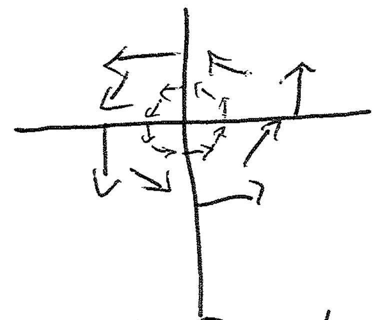
Ex $\vec{F}(x,y) = (-y, +x)$

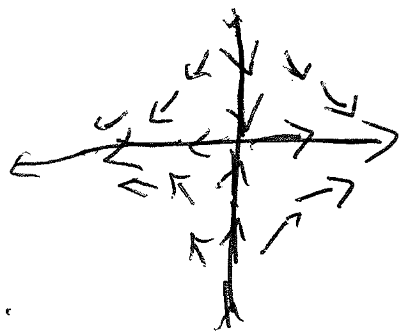


The "flow lines" are circles.

Region of gas rotated, not compressed. Expect $\operatorname{div} \vec{F} = 0$.

$$\operatorname{div} \vec{F} = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x) = 0 + 0 = 0.$$

Ex $\vec{F}(x,y) = (x, -y)$



"Flow lines" are hyperbolas

(level curves of $g(x,y) = xy$).

Compression/Expansion less clear.

$$\operatorname{div} \vec{F} = \frac{\partial}{\partial x} x + \frac{\partial}{\partial y}(-y) = 1 + (-1) = 0.$$

For one branch of hyperbola $xy = c$,
use parametrization $\vec{r}(t) = (e^{t}, c e^{-t})$.

Flow lines solutions to <u>differential equations</u>.

If $\operatorname{div}(\vec{F})(x,y) > 0$, point (x,y) called a "source"

If $\operatorname{div}(\vec{F})(x,y) < 0$, point (x,y) called a "sink".

How to think about divergence theorem for curves:

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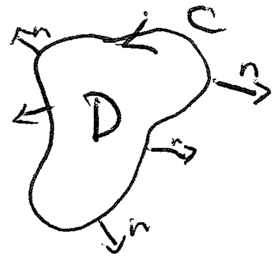
$$\int_{\partial D} F\cdot\vec{n}\,ds = \iint_{D} \operatorname{div}\vec{F}\,dA.$$

Flux

amount of fluid escaping D through C



infinite sum of all sources (expansion → escape) minus sum of all sinks (compression → retention)

The Divergence Theorem for surfaces (aka Gauss' Theorem)

Surface S = boundary of region E in $\mathbb{R}^3$ must be "nice enough"

Assume components of $\vec{F}$ have continuous partials.

Then
$$\iint_{S} F\cdot dS = \iiint_{E} \operatorname{div}\vec{F}\,dV.$$

Same intuition: amount of fluid pushed across boundary determined by compression/expansion inside E.

(Using $\vec{n}$ pointing outwards to compute Flux).

Why? Write $F = P\vec{i} + Q\vec{j} + R\vec{k}$.

Then
$$\text{LHS} = \iint_{S} (P\vec{i} + Q\vec{j} + R\vec{k})\cdot\vec{n}\,dS$$
$$= \iint_{S} P(\vec{i}\cdot\vec{n})\,dS + \iint_{S} Q(\vec{j}\cdot\vec{n})\,dS + \iint_{S} R(\vec{k}\cdot\vec{n})\,dS.$$

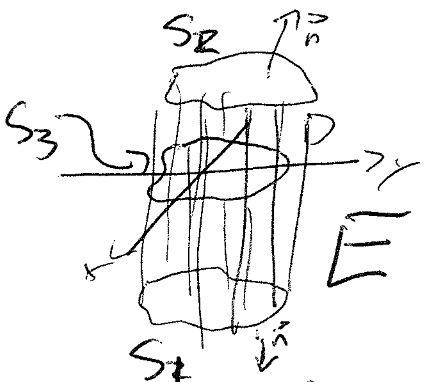
$$\text{RHS} = \iiint_{E} \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\,dV$$

So enough to identify 3 pieces from each equation. For example, want

$$\iint_S R(\vec{k}\cdot\vec{n})\,dS = \iiint_E \frac{\partial R}{\partial z}\,dV.$$

Suppose E nice: "top" S_2 of E described by $z = f_2(x,y)$,
"bottom" S_1 " " $z = f_1(x,y)$.

Then
$$\iiint_E \frac{\partial R}{\partial z}\,dV = \iint_D \int_{f_1}^{f_2} \frac{\partial R}{\partial z}\,dz\,dx\,dy$$



$$= \iint_D R(x,y,f_2) - R(x,y,f_1)\,dx\,dy.$$

On the other hand,

$$\iint_S R(\vec{k}\cdot\vec{n})\,dS = \iint_{S_1} R(\vec{k}\cdot\vec{n})\,dS + \iint_{S_2} R(\vec{k}\cdot\vec{n})\,dS$$

$$\cancel{+\iint_{S_3} R(\vec{k}\cdot\vec{n})\,dS}.$$

On S_3 $\vec{k}\cdot\vec{n} = 0$ $\Longrightarrow$

On S_1, $\vec{n} = \dfrac{\left(\frac{\partial f_1}{\partial x}, \frac{\partial f_1}{\partial y}, -1\right)}{\|\;\;\|}$

so $\vec{k}\cdot\vec{n} = \dfrac{-1}{\sqrt{\left[\frac{\partial f_1}{\partial x}\right]^2 + \left[\frac{\partial f_1}{\partial y}\right]^2 + 1}}$ $\leftarrow \|\vec{r}_x\times\vec{r}_y\|$

$$\iint_{S_1} R\cdot\frac{-1}{\|\vec{r}_x\times\vec{r}_y\|}\,dS = \iint_D R(x,y,f_1)\frac{-1}{\|\vec{r}_x\times\vec{r}_y\|}\|\vec{r}_x\times\vec{r}_y\|\,dA$$

$$= -\iint_D R(x,y,f_1)\,dA.$$

Same idea shows $\iint_{S_2} R\, \vec{k}\circ\vec{n}\, dS = \iint_D R(x,y,h)\, dA$

Ex $F = (2x, y^2, z^2)$, S unit sphere.

Find flux.

$$\iint_S F \circ dS = \iiint \operatorname{div} F \, dV = \iiint_E 2 + 2y + 2z \, dV.$$

$E =$ Solid sphere

$$= 2\iiint_E dV + 2\iiint_E y\, dV + 2\iiint_E z\, dV.$$

$$= 2\operatorname{Vol}(E) = 2 \circ \tfrac{4}{3}\pi = \tfrac{8}{3}\pi.$$

0 since region E symmetric across $y=0$ or $z=0$ planes.

<u>Indeed</u> $\iiint_E y\, dV = \int_{-1}^{1}\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}}\left(\int_{-\sqrt{1-x^2-z^2}}^{\sqrt{1-x^2-z^2}} y\, dy\right) dx\, dz$

$\quad\quad = 0$ since y is <u>odd</u>.

Topics for after <u>Thanksgiving</u>:

curl, conservative vector fields in $\mathbb{R}^3$, Stokes's Theorem