Lecture 36

Nov. 28 2011

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Conflict Final: contact me ASAP

Quiz: Thursday (on Flux, Divergence Theorem)

Before the break:

Divergence Theorem: $S = \partial E$, $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$,

$\vec{F} = (P, Q, R)$ have continuous partials

Then $$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F} \, dV$$

Ex $\vec{F} = (2x, y^2, z^2)$ S unit sphere.

Find flux of $\vec{F}$ across S.

$$\text{Flux} = \iint_S \vec{F} \cdot d\vec{S} \underset{\text{Div Thm}}{=} \iiint_E \operatorname{div} \vec{F} \, dV$$

$$= \iiint_E 2 + 2y + 2z \, dV$$

$$= 2\iiint_E dV + 2\iiint_E y \, dV + 2\iiint_E z \, dV$$

$$= 2 \operatorname{Vol}(E) + 0 + 0$$ "zero by symmetry"

$$= \frac{8\pi}{3}$$

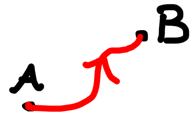
$$\iiint_E y \, dV = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \underbrace{\int_{-\sqrt{1-x^2-z^2}}^{\sqrt{1-x^2-z^2}} y \, dy}_{=0} \, dz \, dx$$

This week: Stokes' Theorem & Curl

Review previous integration theorems

1) For $f: \mathbb{R}^1 \to \mathbb{R}^1$ FTC $[a,b]$ = curve in $\mathbb{R}^1$

$$f(b) - f(a) = \int_a^b f'(x)\,dx$$

2) $f: \mathbb{R}^2 \to \mathbb{R}^1$ or $\mathbb{R}^3 \to \mathbb{R}^1$, C curve in  in $\mathbb{R}^2$ or $\mathbb{R}^3$

FTC for line integrals: $f(B) - f(A) = \int_C \nabla(f) \cdot d\vec{r}$

3) $F: \mathbb{R}^2 \to \mathbb{R}^2$, D surface in $\mathbb{R}^2$ (domain)

Green $\int_{\partial D} \vec{F} \cdot d\vec{r} = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$

4) $F: \mathbb{R}^3 \to \mathbb{R}^3$, S surface in $\mathbb{R}^3$, $C = \partial S$

?? $\int_C \vec{F} \cdot d\vec{r} = \iint_S \,?\, dS$

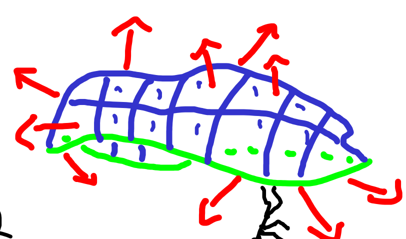
<u>Stokes' Theorem</u>: $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$ $\vec{F} = (P, Q, R)$

S orientable surface, $C = \partial S$ Have continuous partials

Then
$$\boxed{\int_C \vec{F} \cdot d\vec{r} = \iint_S (\operatorname{curl} \vec{F}) \cdot dS}$$

<u>Note</u>: Orientation implicit on RHS (choice of $\vec{n}$).
How is this reflected in LHS?

C should be parametrized in direction where S appears on left if walking with head pointing in direction $\vec{n}$.



What is "curl $\vec{F}$"?

From §16.5:

$$\text{curl } \vec{F} \text{ "="} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \longleftarrow \text{"} \nabla \times F \text{"}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)$$

Note: curl is only defined for vector fields in $\mathbb{R}^3$.

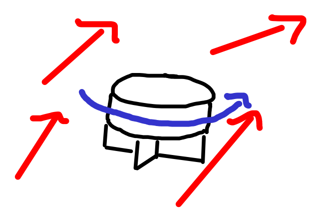
Ex $\vec{F}(x,y,z) = (x,y,z)$

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \vec{0}$$

Ex $\vec{F}(x,y,z) = (-y, x, 1)$.

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 1 \end{vmatrix} = (0, 0, 2)$$

What does this measure?



Imagine small paddle wheel moving according to $\vec{F}$.

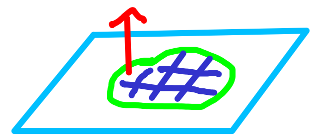
curl $\vec{F}$ measures rotation of wheel as it is moved by $\vec{F}$

(curl $\vec{F}$ points in direction of axis of rotation & magnitude gives angular speed)

Ex 1 has no rotation but Ex 2 has rotation in z-direction.

Back to Stokes' Theorem

Suppose $R=0$ and S contained in $z=0$ plane.



Then

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = \left(-\frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

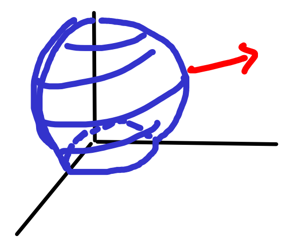
Take $\vec{n} = \vec{k}$. Then Stokes says

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}(\vec{F})\cdot \vec{n}\, dS$$
$$= \iint_S \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dS$$

This is Green's theorem!

Ex S described by $x^2+y^2+(z-1)^2 = 2$, $z \geq 0$

$\vec{F} = (-y, x, 1)$. ∂S = unit circle



Find $\iint_S (\nabla \times F)\cdot dS$

By Stokes, this is $\int_C \vec{F}\cdot d\vec{r}$ ← use $\vec{r}(t) = (\cos t, \sin t, 0)$

$$= \int_0^{2\pi} (-\sin t, \cos t, 1)\cdot(-\sin t, \cos t, 0)\, dt$$
$$= \int_0^{2\pi} \sin^2 t + \cos^2 t\, dt = 2\pi.$$

Let S_2 = rest of sphere (so $z \leq 0$). Same boundary, so should also get $\iint_{S_2} (\nabla\times F)\cdot dS = 2\pi$.

Compute directly: use param.

$$\vec{r}(\phi,\theta) = (\sqrt{2}\cos\theta\sin\phi, \sqrt{2}\sin\theta\sin\phi, \sqrt{2}\cos\phi + 1)$$

$$\frac{3\pi}{4} \le \phi \le \pi, \quad 0 \le \theta \le 2\pi$$

$$\nabla \times F = (0, 0, 2)$$

$$\vec{r}_\phi \times \vec{r}_\theta = (2\cos\theta \sin^2\phi, 2\sin\theta \sin^2\phi, 2\cos\phi \sin\phi)$$

$$(\nabla \times F) \cdot \vec{n} = 4\cos\phi \sin\phi$$

$$\iint_{S_2} (\nabla \times F) \cdot dS = \int_0^{2\pi} \int_{3\pi/4}^{\pi} 4\cos\phi \sin\phi \, d\phi \, d\theta$$

$$= 4 \cdot 2\pi \left. \frac{\sin^2\phi}{2} \right|_{3\pi/4}^{\pi}$$

$$= -2\pi$$

why?

Because C gets the clockwise parametrization as the boundary of S_2.