Lecture 38

Dec 1 2011

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Recall that $\vec{F}$ is <u>conservative</u> if $F = \nabla(f)$, some f.

For $\vec{F}: \mathbb{R}^2 \to \mathbb{R}^2$, had

<u>1st Recognition Thm</u> $\vec{F}$ conservative on connected domain

$\iff \oint_C \vec{F} \cdot d\vec{r} = 0$ for all closed loops

<u>2nd Recognition Thm</u> F conservative on simply connected domain

$\iff \dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$. $(F = (P, Q))$.

1st thm works just as well for $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$

<u>Today</u>: will give version of 2nd thm for $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$.

First, review why 2nd thm holds. By Green,

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_D \underbrace{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}_{=0 \text{ by Assumption}} \, dA = 0$$

, so done by 1st theorem.

C closed loop, domain of $\vec{F}$ has no holes $\Rightarrow$ C is the boundary of some region D.

$\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}$ is one term in curl $\vec{F}$. What is curl $\vec{F}$ if $\vec{F} = \nabla f$?

$$\nabla \times \nabla f = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}, \frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}, \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right)$$

$= 0$ by Clairaut

So $\operatorname{curl}(\nabla f) = \vec{0}$

By the way, $\operatorname{curl} \vec{F} = $ "$\nabla \times \vec{F}$", so $\operatorname{curl} \nabla f = \nabla \times (\nabla f)$

$$= (\nabla \times \nabla) f = \vec{0}$$

Ex $F(x,y,z) = (y, z, x)$.

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & z & x \end{vmatrix} = (0-1, -1+0, 0-1) = (-1,-1,-1)$$

$\nabla \times \vec{F} \neq \vec{0}$, so $\vec{F}$ cannot be conservative.

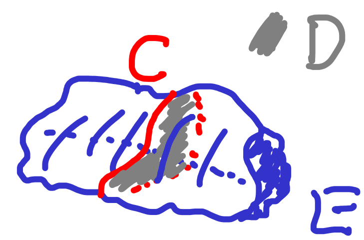
2nd Recognition Thm for vector fields of 3 variables:

$\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$ defined on simply-conn region.

$\vec{F}$ conservative $\Longleftrightarrow \nabla \times \vec{F} = \vec{0}$.

Why? Again, by 1st thm, enough to show

$\int_C \vec{F} \cdot d\vec{r} = 0$ for all closed loops C.



Region simply conn $\Rightarrow$ can "fill in" loop C by disk D contained in E.

Then $\int_C \vec{F} \cdot d\vec{r} \overset{\text{Stokes}}{=} \iint_D \underbrace{(\nabla \times \vec{F})}_{=0} \cdot dS = 0$.

Recall also had $\operatorname{div} \vec{F} = \nabla \cdot \vec{F}$.

So $\operatorname{div}(\operatorname{curl}\vec{F}) = \nabla\cdot(\nabla\times\vec{F})$ ← "triple product"

$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix}$$

$$= \begin{vmatrix} \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} = \underline{\underline{0}}$$ since 2 rows are the same.

This allows us to test whether a vector field is a curl field: $\vec{F} = (x, y, z)$. Then

$\operatorname{div}\vec{F} = 1 + 1 + 1 = 3 \neq 0$, so $\vec{F}$ is <u>not</u> $\nabla\times\vec{G}$ for some $\vec{G}$.

This means $\iint_S \vec{F}\cdot dS$ does not (<u>just</u>) depend on ∂S.

<u>Ex</u> S_1 = northern unit hemisphere, S_2 = unit disc in x,y plane.

$$\iint_{S_1} \vec{F}\cdot dS = \iint_{S_1} \underset{\vec{F}}{(x,y,z)}\cdot\underset{\vec{n}}{(x,y,z)}\, dS$$

$$= \iint_{S_1} \underbrace{x^2+y^2+z^2}_{=1}\, dS = \iint_{S_1} dS = \frac{2\pi}{3}.$$

$$\iint_{S_2} \vec{F}\cdot dS = \iint_{S_2} (x,y,z)\cdot(0,0,1)\, dS$$

$$= \iint_{S_2} z\, dS = 0.$$

$z = 0$ on S_2.

Note $\partial S_1 = \partial S_2 =$ unit circle C

If we had $\vec{F} = \nabla \times \vec{G}$, then

$$\iint_{S_1} \vec{F} \cdot dS = \iint_{S_1} (\nabla \times \vec{G}) \cdot dS \underset{\text{Stokes}}{=} \int_C \vec{G} \cdot d\vec{r} \underset{\text{Stokes}}{=} \iint_{S_2} \vec{F} \cdot dS$$

We know $\operatorname{div}(\nabla \times \vec{F}) = 0$.

Q If $\operatorname{div} \vec{F} = 0$, is $\vec{F} = \nabla \times \vec{G}$ some $\vec{G}$?

A If $\vec{F}$ defined on all of $\mathbb{R}^3$, then YES.

Let $\vec{G} = \left[\int_0^z Q\,dt - \int_0^y R\,dt\right]\vec{i} - \left(\int_0^z P\,dt\right)\vec{j}$.

Then

$$\nabla \times \vec{G} = \frac{\partial}{\partial z}\int_0^z P\,dt\,\vec{i} + \frac{\partial}{\partial z}\left[\quad\right]\vec{j} + \frac{\partial}{\partial x}\left[-\int_0^z P\,dt\right] - \frac{\partial}{\partial y}\left[\quad\right]\vec{k}$$

$$= P\vec{i} + Q\vec{j} + R\vec{k} = \vec{F}$$

In summary:

$$\begin{tikzcd}
\boxed{f: \mathbb{R}^3 \to \mathbb{R}} \arrow[r, "\nabla"] \arrow[rr, bend right, "\vec{0}"] & \boxed{F: \mathbb{R}^3 \to \mathbb{R}^3} \arrow[r, "\nabla\times"] \arrow[rr, bend right, "0"] & \boxed{G: \mathbb{R}^3 \to \mathbb{R}^3} \arrow[r, "\operatorname{div}"] & \boxed{h: \mathbb{R}^3 \to \mathbb{R}}
\end{tikzcd}$$

If $\nabla \times \vec{F} = 0$, $\vec{F} = \nabla f$ some f

If $\operatorname{div} \vec{G} = 0$ $\vec{G} = \nabla \times \vec{F}$ some $\vec{F}$.

called an exact sequence

No longer "exact" if change domain.