Lecture 39

Dec 5 2011

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This week usual tutoring hours M – Thur
Also Fri & Mon Dec 12 10AM–2PM 159 Altgeld

In summary:

$$\begin{tikzcd}[column sep=large]
\boxed{0} & \boxed{1} & \boxed{2} & \boxed{3} \\
\boxed{f : \mathbb{R}^3 \to \mathbb{R}} \arrow[r, "\substack{\text{grad}\\ \nabla}"] \arrow[rr, bend right, "\vec{0}"'] & \boxed{F : \mathbb{R}^3 \to \mathbb{R}^3} \arrow[r, "\substack{\text{curl}\\ \nabla\times}"] \arrow[rr, bend right, "0"'] & \boxed{G : \mathbb{R}^3 \to \mathbb{R}^3} \arrow[r, "\text{div}"] & \boxed{h : \mathbb{R}^3 \to \mathbb{R}}
\end{tikzcd}$$

If $\nabla \times \vec{F} = 0$, $\vec{F} = \nabla f$ some f
If $\operatorname{div} \vec{G} = 0$ $\vec{G} = \nabla \times \vec{F}$ some $\vec{F}$.

called an exact sequence

No longer "exact" if change domain.

Ex $\vec{G} : \mathbb{R}^3 \setminus \{\vec{0}\} \longrightarrow \mathbb{R}^3$

$$\vec{G}(x,y,z) = \frac{1}{\rho^3}(x,y,z) \qquad \rho = \sqrt{x^2+y^2+z^2}$$

Then $$\frac{\partial}{\partial x}\frac{x}{\rho^3} = \frac{\rho^3 - x\,3\rho^2\frac{2x}{2\rho}}{\rho^6} = \frac{\rho^2 - 3x^2}{\rho^5}$$

$$\frac{\partial}{\partial x}\frac{x}{\rho^3} + \frac{\partial}{\partial y}\frac{y}{\rho^3} + \frac{\partial}{\partial z}\frac{z}{\rho^3} = \frac{3\rho^2 - 3[x^2+y^2+z^2]}{\rho^5} = 0.$$

So $\operatorname{div}\vec{G} = 0$. But $\vec{G} \neq \nabla \times \vec{F}$.

S_1 = unit sphere Note $\rho = 1$ on S_1.

$$\iint_{S_1} \frac{1}{\rho^3}(x,y,z)\cdot d\vec{S} = \iint_{S_1} 1\, dS = 4\pi.$$

By Stokes, if $\vec{G} = \nabla \times \vec{F}$, then $\iint_{S_1} \vec{G} \cdot d\vec{S}$ should be 0

since S_1 has empty boundary.

So for domain $\mathbb{R}^3 \setminus \{\vec{0}\}$, the sequence is not "exact" at position $\boxed{2}$.

Since $\mathbb{R}^3 \setminus \{0\}$ is simply-connected, it is still "exact" at position $\boxed{1}$

Ex Similarly, define $\vec{F} : \mathbb{R}^3 \setminus z\text{-axis} \to \mathbb{R}^3$

$$\vec{F}(x,y,z) = \frac{1}{x^2+y^2}(-y, x, 0).$$

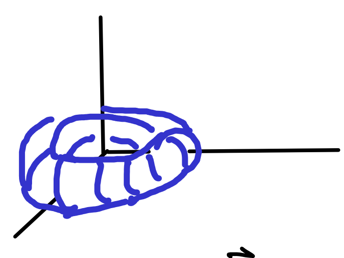
Saw previously that $\nabla \times \vec{F} = \vec{0}$ but $\vec{F}$ is not conservative, since

$$\int_C \vec{F} \cdot d\vec{r} = 2\pi \neq 0.$$

unit circle. So the sequence is not "exact" at position $\boxed{1}$.

It turns out it is exact at position $\boxed{2}$.

Ex Torus $(r-2)^2 + z^2 = 1$



Not exact at position $\boxed{1}$ or $\boxed{2}$.

Again, $\vec{F} = \frac{1}{r^2}(-y, x, 0)$ defined on T, $\nabla \times \vec{F} = 0$ but $\vec{F}$ not conservative.

Can say more:

For $\mathbb{R}^3 \setminus z$-axis, the only fields $\vec{F}$ satisfying $\nabla \times \vec{F} = \vec{0}$, $\vec{F}$ not conservative are

$$\vec{F} = \frac{c}{x^2+y^2}(-y, x, 0).$$

constant

For $\mathbb{R}^3 \setminus z$-axis and line $x=1, y=0$ the only such fields are $\vec{F}_1 = \frac{c}{x^2+y^2}(-y, x, 0)$

and $$\vec{F}_2 = \frac{d}{(x-1)^2+y^2}(-y, x-1, 0)$$

or any combinations of $\vec{F}_1$, $\vec{F}_2$.

<u>Moral</u>: The grad-curl-div sequence knows about the "shape" of the domain.

The general Stokes theorem.

<u>Terminology</u> "0-form" = function

"differential forms" → "1-form" something like $f\,dx - g\,dz$ or ds

→ "2-form" something like $f\,dx\,dy + g\,dy\,dz$ or dA or dS

→ "3-form" something like $f\,dx\,dy\,dz$ or dV.

There are operators ("the differential")

$$\begin{tikzcd} \{0\text{-forms}\} \arrow[r, "d^0"] \arrow[rr, bend right, "0"'] & \{1\text{-forms}\} \arrow[r, "d^1"] \arrow[rr, bend right, "0"'] & \{2\text{-forms}\} \arrow[r, "d^2"] & \{3\text{-forms}\} \end{tikzcd}$$

defined as follows

$$d^0(f) = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

$$= \nabla(f) \cdot (dx, dy, dz).$$

$$d^1(f\,dx_i) = \frac{\partial f}{\partial x} dx\, dx_i + \frac{\partial f}{\partial y} dy\, dx_i \; \frac{\partial f}{\partial z} dz\, dx_i$$

$$d^2(f\,dx_i\,dx_j) = \frac{\partial f}{\partial x} dx\, dx_i\, dx_j + \cdots + \frac{\partial f}{\partial z} dz\, dx_i\, dx_j.$$

where 1) each d^j is <u>linear</u>

2) $dx_i\,dx_j = -dx_j\,dx_i$

3) $dx\,dx = 0$.

<u>Thm</u> (General Stokes) M any nice region ("manifold")

ω (omega) any diff. form.

Then $\displaystyle\int_M d\omega = \int_{\partial M} \omega$.

<u>Ex</u> Green's thm

$$\int_{\partial D} \vec{F} \cdot d\vec{r} = \int_{\partial D} P\,dx + Q\,dy \overset{\text{Green}}{=} \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dx\,dy.$$

$\omega = P\,dx + Q\,dy$. Then

$$d\omega = d(P\,dx + Q\,dy) = d(P\,dx) + d(Q\,dy)$$

$$= \frac{\partial P}{\partial x} \underbrace{dx\,dx}_{0} + \frac{\partial P}{\partial y} dy\,dx + \frac{\partial Q}{\partial x} dx\,dy + \frac{\partial Q}{\partial y} \underbrace{dy\,dy}_{0}$$

$$= \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\; dx\,dy$$

Ex Stokes' thm

$$\int_{\partial S} \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot dS$$

Here $\omega = P\,dx + Q\,dy + R\,dz$

and we find $d\omega = \frac{\partial P}{\partial y}\, dy\,dx + \frac{\partial P}{\partial z}\, dz\,dx + \cdots$

$$= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy + \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) dx\,dz$$

$$+ \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy\,dz$$

Fact If $\vec{n} = (n_1, n_2, n_3)$ is outward pointing normal,

then $dx\,dy = n_3\,dS$, $dz\,dx = n_2\,dS$, $dy\,dz = n_1\,dS$

So $d\omega = (\nabla \times \vec{F}) \cdot \vec{n}\, dS$

Ex Div Thm $\displaystyle \iint_{\partial E} \vec{F} \cdot \vec{n}\, dS = \iiint_E \operatorname{div} F\, dV$

$\omega = \vec{F} \cdot \vec{n}\, dS = P\,n_1\,dS + Q\,n_2\,dS + R\,n_3\,dS$

$$= P\,dy\,dz + Q\,dx\,dz + R\,dx\,dy$$

So $d\omega = \frac{\partial P}{\partial x} dx\,dy\,dz + \frac{\partial Q}{\partial y} dy\,dz\,dx + \frac{\partial R}{\partial z} dz\,dx\,dy$

$$= \operatorname{div} \vec{F}\, dx\,dy\,dz.$$